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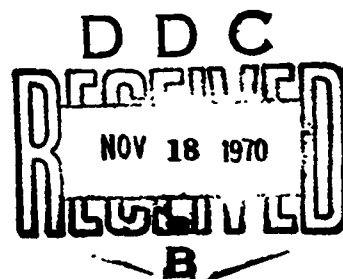
SUBOPTIMIZATION OF A KALMAN FILTER WITH DELAYED-STATES AS OBSERVABLES

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**TECHNICAL REPORT
SUBOPTIMIZATION OF A
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AS OBSERVABLES**

**Terry B. Cline
C. James Triska**

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I. INTRODUCTION

The problem of the excessive computational requirements of the Kalman filter has been studied extensively in recent years. For most practical applications, a solution of this problem is the critical factor in deciding whether or not the filter can be implemented in an operational system.

The problem becomes more acute in the case of a delayed-state Kalman filter due to the increased complexity of the filter equations introduced by accounting for the connection of the measurement with the previous state.

One approach to the solution of the computational problem has been to reduce the dimension of the state and estimate only those state variables that are of prime interest; see Levy [14], Schmidt and Lukesh [25], Simon and Stubberud [27]. The state variables often neglected in this reduced filter method are the correlated noise processes in the system. They are, in effect, modeled as uncorrelated noise processes when this method is used. Although optimal, the method proposed by Simon and Stubberud [27] is subject to the restriction that the reduction in the dimension of the state is limited by the dimension of the measurement vector.

In another approach, Aoki and Huddle [3] constructed a so-called minimal order observer, utilizing the observer theory of Luenberger [15], which was composed of two distinct sections. The first represented a

dynamic subsystem of the observed system whose output, under proper conditions, was a linear transformation of the observed system state vector. An estimate of the observed system state vector was constructed by minimizing the elements of the estimation error covariance matrix. The method requires proper specification of several matrices for the steady-state condition of the filter.

Brock and Schmidt [4] proposed using preset or precomputed gains based upon the high and low frequency response of the system in an inertial navigation application. Sims and Melsa [28] investigated the effect of using constant and exponential gain functions, which could be precomputed, based on the results of simulation studies. Precomputation of gains suffers from the requirement of a priori knowledge of the system's behavior in order to predict adequately the form the gain should take.

The method to be applied here to the delayed-state filter case is due to Joseph [12] and was formalized by Pentecost [22]. The advantage of this method is that the computational requirements are significantly reduced, by partitioning of the system into lower dimensional subsystems, with only minimum a priori knowledge of the system's behavior; i.e., the gains are computed on-line.

The degradation in the performance of a continuous Kalman filter introduced by the use of a continuous reduced-order filter has been studied by Huddle and Wismer [11]. Nishimura [16] performed a sensitivity analysis on the continuous Kalman filter performance for inaccuracies

in the specification of the noise covariance matrices and of the initial state covariance matrix.

Griffin and Sage [9] studied the continuous and discrete Kalman filter and the continuous smoothing equations for sensitivity to modeling and noise covariance matrix errors analogous to our method for studying the performance of a suboptimal discrete-time delayed-state filter due to simplified modeling. Large and small scale sensitivity indices were defined also, which are not useful for our situation as they are only practical for a small number of parameter changes.

Heffes [10] studied the sensitivity of the gain computation to modeling errors in the discrete Kalman filter and Sorensen [29] determined bounds on the estimation error covariance matrix of the optimal Kalman filter.

Price [24] derived the recursive equations for the suboptimal estimation error covariance matrix of the discrete-time Kalman filter using a perturbed model by a method equivalent to the one used here for the delayed-state filter. He then determined bounds on the suboptimal estimation error covariance matrix and conditions on the model changes under which it was asymptotically stable in the large, using Lyapunov theory techniques. The determination of such bounds and conditions for stability are not considered here for the delayed-state filter. A method similar to Price's could be used for such a purpose.

We here derive expressions for the estimation error covariance matrix for suboptimizations of the delayed-state filter based on

modeling simplifications and alternate gain computations which place less stringent computational requirements on the system computer than does the optimal filter.

A method based upon the work of Pentecost [22] is presented for the reduction of computational requirements of the delayed-state filter.

A performance index is defined for use as a relative measure of the performance of the various suboptimizations compared to the optimal filter. The results of a simulation of an integrated inertial navigation system are presented as an application of the theoretical results.

II. THE DELAYED-STATE KALMAN FILTER

As previously noted, measurements which are linearly connected to not only the present state but the previous state as well require a modification of the usual Kalman equations. This was first done by Brown and Hartman [7]. The purpose of this chapter is to summarize the resulting equations and introduce the notation that will be used in the sequel. The derivation of the equations will be presented in the appendix.

The random process to be estimated is assumed to be a linear system satisfying the vector differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{w}(t) \quad (2.1)$$

where $\mathbf{x}(t)$ is the n -dimensional state vector at time t , $\mathbf{A}(t)$ is an $n \times n$ time-varying matrix at time t , $\mathbf{w}(t)$ is a p -dimensional white noise input vector, and $\mathbf{B}(t)$ is the $n \times p$ time-varying matrix which connects the input to the state at time t .

Solution of (2.1) results in, for the discrete-time case,

$$\mathbf{x}(k+1) = \Phi(k+1,k)\mathbf{x}(k) + \mathbf{u}(k) \quad (2.2)$$

where $\Phi(k+1,k)$ is the $n \times n$ state transition matrix from stage k to stage $k+1$, and $\{\mathbf{u}(k)\}$, the plant noise, is a white noise sequence of n -dimensional vectors.

The measurement model is assumed to be of the form

$$y(k+1) = M(k+1)x(k+1) + N(k+1)x(k) + v(k+1) \quad (2.3)$$

where $y(k+1)$ is an m -dimensional measurement vector, $M(k+1)$ is an $m \times n$ time-varying matrix which linearly connects the present state to the measurement, $N(k+1)$ is an $m \times n$ time-varying matrix which linearly connects the previous state to the measurement, and $\{v(k+1)\}$ is an m -dimensional white noise sequence, the measurement noise.

The following statistical properties of the system (2.2), (2.3) are assumed:

- 1) $\{x(k), k = 0, 1, \dots\}$ and $\{y(j), j = 1, 2, \dots\}$ are gaussian with zero means.
- 2) $\{u(k), k = 0, 1, \dots\}$ and $\{v(j), j = 1, 2, \dots\}$ are independent gaussian white noise with zero mean.
- 3) $E\{x(j)u'(k)\} = 0$ for all $k \geq j$, $j = 0, 1, \dots$, where $E\{\cdot\}$ indicates the expectation operator and the prime denotes the transpose.
- 4) $E\{y(j)u'(k)\} = 0$ for all $k \geq j + 1$, $j = 0, 1, \dots$
- 5) $E\{x(j)v'(k)\} = 0$ for all j, k , $j = 0, 1, \dots$, $k = 1, 2, \dots$
- 6) $E\{y(j)v'(k)\} = 0$ for all $k > j + 1$, $j = 1, 2, \dots$
- 7) $E\{v(j)v'(k)\} = V(k)\delta_{jk}$, $j, k = 1, 2, \dots$, where δ_{jk} is the Kronecker delta function and $V(k)$ is a positive semidefinite $m \times m$ matrix.
- 8) $E\{u(j)u'(k)\} = H(k)\delta_{jk}$, $j, k = 0, 1, \dots$, where $H(k)$ is a positive semidefinite $n \times n$ matrix.

The recursive equations for estimating the state of the model (2.2)-(2.3) at stage k can be summarized from Brown and Hartman as

$$\begin{aligned} Q(k) = & M(k)P(k|k-1)M'(k) + V(k) + N(k)P(k-1|k-1)N'(k) \\ & + M(k)\Phi(k,k-1)P(k-1|k-1)N'(k) \\ & + [M(k)\Phi(k,k-1)P(k-1|k-1)N'(k)]' \end{aligned} \quad (2.4)$$

$$K(k) = [P(k|k-1)M'(k) + \Phi(k,k-1)P(k-1|k-1)N'(k)]Q^{-1}(k) \quad (2.5)$$

$$P(k|k) = P(k|k-1) - K(k)Q(k)K'(k) \quad (2.6)$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)[y(k) - M(k)\hat{x}(k|k-1) - N(k)\hat{x}(k-1|k-1)] \quad (2.7)$$

$$\hat{x}(k+1|k) = \Phi(k+1,k)\hat{x}(k|k) \quad (2.8)$$

$$P(k+1|k) = \Phi(k+1,k)P(k|k)\Phi'(k+1,k) + H(k) \quad (2.9)$$

where $\hat{x}(k|k)$ is the optimal estimate of the stage at stage k given measurements through stage k and $\hat{x}(k+1|k)$ is the optimal estimate of the state at stage $k+1$ given measurements through only stage k . $K(k)$ is the optimal gain matrix at stage k . $P(k|k)$ is the covariance matrix of the estimation error $\tilde{x}(k|k) = [x(k) - \hat{x}(k|k)]$ and $P(k+1|k)$ is the covariance matrix of the estimation error $\tilde{x}(k+1|k) = [x(k+1) - \hat{x}(k+1|k)]$. $Q(k)$ is the covariance matrix of the measurement estimation error $\tilde{y}(k|k-1) = [y(k) - \hat{y}(k|k-1)]$.

These recursive equations uniquely specify a solution at any time k given the initial conditions, $P(0|0)$, and $\hat{x}(0|0)$. A block diagram

indicating the structure of the delayed-state filter is shown in Fig. 1, where the double lines denote vector signals and the blocks denote matrix operations.

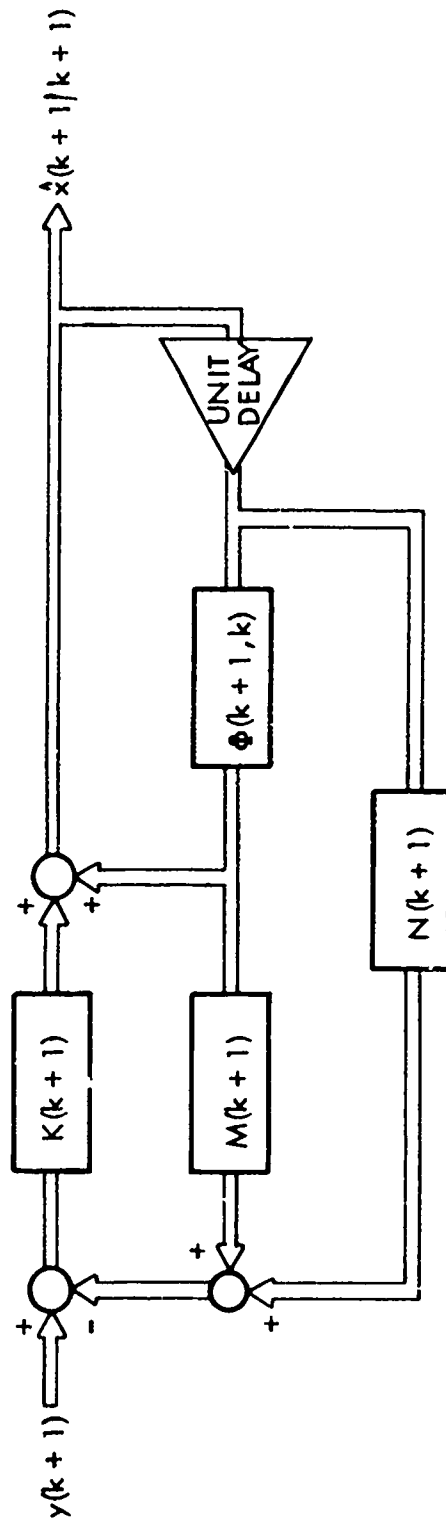


Fig. 1. Structure of the delayed-state filter

III. PERFORMANCE ANALYSIS OF THE DELAYED-STATE FILTER

When an analysis on a dynamical system is performed for which the estimate of the state variables will ultimately be required, the question arises as to how complete the model must be to adequately predict the behavior of the system and yet result in computationally efficient estimation algorithms. As an aid in performing such an analysis, it is useful to have an algorithm for determining the degradation in performance incurred in using a simplified model of the system dynamics.

Also, since the gain computation in the Kalman filter places the most severe requirements on the computational abilities of the computer in an operational system, it is sometimes desirable to use suboptimal gains which are easier to compute. Thus, it is desirable to have an algorithm to evaluate the degradation of the estimates as compared to those estimates achieved with the optimal gain computation.

The purpose of this chapter is to derive algorithms which accomplish both of the above purposes. The method of attack for the simplified model case is analogous to, but different than, the method used by Price [24]. For use as a figure of merit for the various models and suboptimal filters that may be used, a performance index will be defined. The effect of modeling differences in the noise covariances are not included here but could easily be included by suitable modification of the equations that follow.

A. Performance Analysis of a Suboptimal Filter Using

A Simplified Model

Suppose that a change is made in the modeling of the system dynamics for some reason, such as reducing small terms to zero or making simplifying assumptions which result in changes in some terms or reducing the number of state variables. The result is a perturbation in the $A(t)$ matrix of the system model (2.1)-(2.3) which introduces a perturbation in the filter equations, (2.4)-(2.9). If the original model is assumed to be the most complete and correct one possible, it will result in optimal estimates by the filter. Thus, if the model is changed or simplified, suboptimal estimates will result. We will call this model the design model.

Let the original system model be described by (from chapter II)

$$x(k+1) = \Phi(k+1, k)x(k) + u(k) \quad (3.1)$$

$$y(k+1) = M(k+1)x(k+1) + N(k+1)x(k) + v(k+1)$$

and the design system model by

$$x_d(k+1) = \Phi_d(k+1, k)x_d(k) + u(k) \quad (3.2)$$

$$y_d(k+1) = M(k+1)x_d(k+1) + N(k+1)x_d(k) + v(k+1)$$

where $\Phi_d = \Phi + \delta\Phi$. The perturbation in the original system transition matrix $\delta\Phi$ results from computing the transition matrix Φ_d using the design system model, $A(t) + \delta A(t)$. In general, there is no simple relationship between $\delta A(t)$ and $\delta\Phi(k+1, k)$ and one must compute both

transition matrices to get the value of $\hat{\Phi}(k+1, k)$ at any stage k .

By using the design system model in the optimal filter equations, we obtain an optimal estimate \hat{x}_d of x_d , if y_d were the available measurement. The estimation error covariance matrix for estimating x_d given y_d is

$$P_d(k|k) = E\left\{\left[x_d(k) - \hat{x}_d(k|k)\right]\left[x_d(k) - \hat{x}_d(k|k)\right]'\right\}. \quad (3.3)$$

We wish to use the gain obtained from the design system optimal filter and the design system transition matrix to obtain a (suboptimal) estimate of x . The estimator is of the form

$$\hat{x}_s(k|k) = \hat{x}_s(k|k-1) + K_d(k)[y(k) - \hat{y}_d(k|k-1)]. \quad (3.4)$$

The only difference between this estimator and the one for the optimal filter for the design system is that y is used instead of y_d since y_d is not physically available. $\hat{x}_s(k|k-1)$ is computed as $\hat{\Phi}_d(k, k-1)\hat{x}_s(k-1|k-1)$.

The P_d used in computing K_d is not the estimation error covariance matrix for the suboptimal filter, but is the estimation error covariance matrix for estimating x_d . The estimation error covariance matrix for the optimal filter for the original system, where $\hat{\Phi}$ and y are used in the filter equations of chapter II, is

$$P(k|k) = E\left\{\left[x(k) - \hat{x}(k|k)\right]\left[x(k) - \hat{x}(k|k)\right]'\right\}. \quad (3.5)$$

The estimation error covariance matrix for the suboptimal filter, using

Φ_d in the optimal gain computation and in the estimator with y , is

$$P_s(k|k) = E\left\{[x(k) - \hat{x}_s(k|k)][x(k) - \hat{x}_s(k|k)]'\right\} \quad (3.6)$$

To evaluate any degradation in performance, we wish to compare P and P_s . Thus an expression for P_s is required. The relationship between the filters mentioned above is represented schematically in Fig. 2.

In order to evaluate P_s , we note the following:

$$\hat{x}_s(k+1|k) = \Phi(k+1, k)\hat{x}_s(k|k) + \epsilon\Phi(k+1, k)\hat{x}_s(k|k), \quad (3.7)$$

$$\tilde{x}(k+1|k) = \Phi(k+1, k)\tilde{x}_s(k|k) - \delta\Phi(k+1, k)\hat{x}_s(k|k) + u(k), \quad (3.8)$$

$$\begin{aligned} \tilde{x}_s(k+1|k+1) &= \Phi(k+1, k)\tilde{x}_s(k|k) - \delta\Phi(k+1, k)\hat{x}_s(k|k) + u(k) \\ &\quad - K_d(k+1)\tilde{y}_s(k+1|k). \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} \tilde{y}_s(k+1|k) &= y(k+1) - \hat{y}(k+1|k) \\ &= [M(k+1)\Phi(k+1, k) + N(k+1)]\tilde{x}_s(k|k) \\ &\quad - M(k+1)\delta\Phi(k+1, k)\hat{x}_s(k|k) + v(k+1). \end{aligned} \quad (3.10)$$

With respect to the estimation error covariance matrix derivation, the chief difference between the optimal filter and the suboptimal filter considered here is that \tilde{x} satisfies the original system homogeneous state equation and the \tilde{x}_s does not. As we shall see, considerable complexity results.

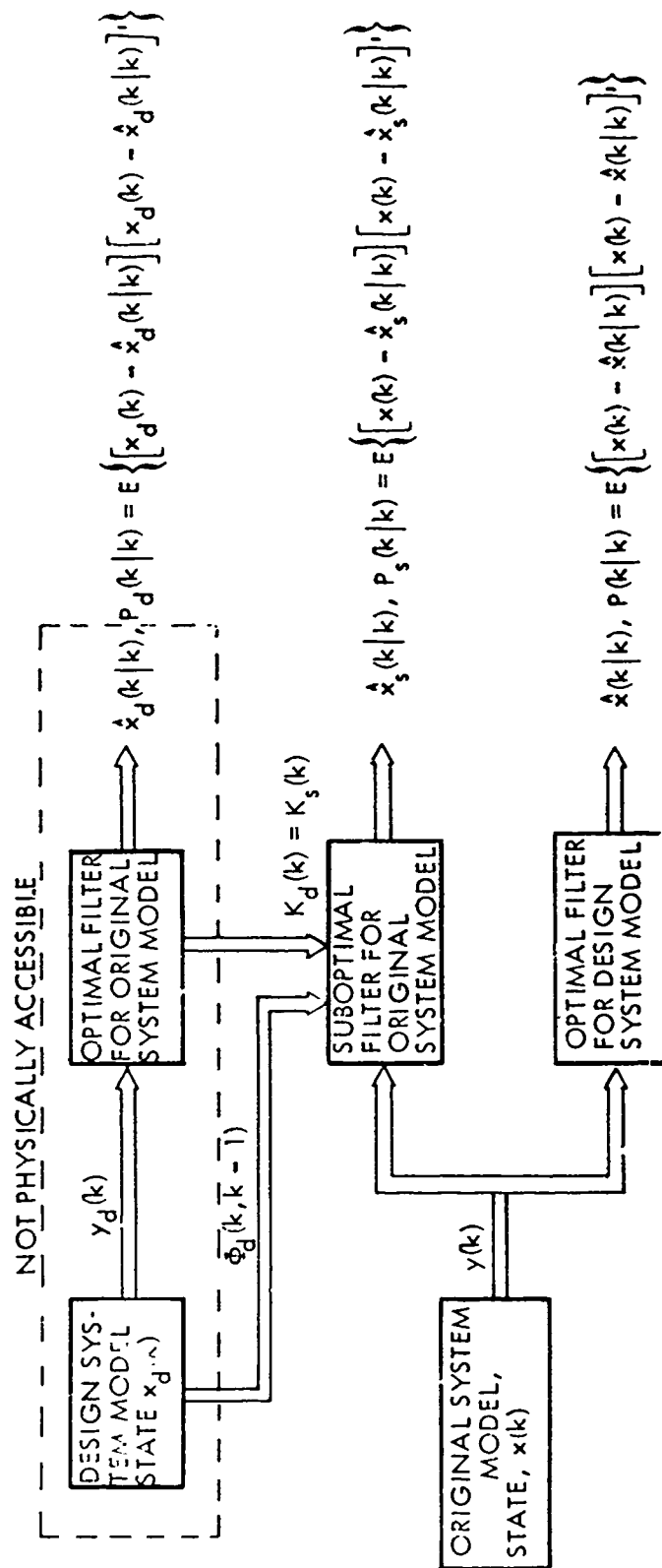


Fig. 2. Conceptual relationship between the suboptimal filter and the optimal filter for the original system model and the optimal filter for the design system model

We first compute the a priori suboptimal estimation error covariance matrix

$$P_S(k+1|k) = E\left\{\tilde{x}_S(k+1|k)\tilde{x}_S'(k+1|k)\right\}.$$

After a great deal of manipulation of (3.7)-(3.10), we obtain

$$\begin{aligned} P_S(k+1|k) &= \Phi(k+1,k)P_S(k|k)\Phi'(k+1,k) \\ &\quad + H(k) + \delta\Phi(k+1,k)S(k)\delta\Phi'(k+1,k) \\ &\quad - \Phi(k+1,k)[R(k) - S(k)]\delta\Phi'(k+1,k) \\ &\quad - \left\{\Phi(k+1,k)[R(k) - S(k)]\delta\Phi'(k+1,k)\right\}'. \end{aligned} \quad (3.11)$$

If second-order effects are neglected,

$$\begin{aligned} P_S(k+1|k) &= \Phi(k+1,k)P_S(k|k)\Phi'(k+1,k) + H(k) \\ &\quad - \delta\Phi(k+1,k)[R(k) - S(k)]\delta\Phi'(k+1,k) \\ &\quad - \left\{\Phi(k+1,k)[R(k) - S(k)]\delta\Phi'(k+1,k)\right\}'. \end{aligned} \quad (3.12)$$

We see that $P_S(k+1|k)$ is of the same form as the a priori estimation error covariance matrix for the optimal filter plus some correction terms due to the change in modeling.

To compute $P_S(k+1|k+1)$, rewrite $\tilde{x}_S(k+1|k+1)$ as

$$\tilde{x}_S(k+1|k+1) = \tilde{x}_S(k+1|k) - K_d(k+1)\tilde{y}_S(k+1|k) \quad (3.13)$$

and $\tilde{y}_s(k+1|k)$ as

$$\tilde{y}_s(k+1|k) = M(k+1)\tilde{x}_s(k+1|k) + N(k+1)\tilde{x}_s(k|k) + v(k+1). \quad (3.14)$$

$$P_s(k+1|k+1) = E\left\{\tilde{x}_s(k+1|k+1)\tilde{x}_s'(k+1|k+1)\right\}$$

Note that $\tilde{x}_s(k+1|k+1)$ and $\tilde{y}_s(k+1|k)$ are of the same form as in the optimal filter. Thus, the first few steps of the derivation are the same as in the optimal filter. The similarity stops when one tries to use the equation for the gain to simplify the equation to the form of equation (2.6). The problem is that the gain is computed using P_d , not P_s , which of course is due to the suboptimization of the filter.

After some routine manipulation,

$$\begin{aligned} P_s(k+1|k+1) = & P_s(k+1|k) - K_d(k+1)[M(k+1)P_s(k+1|k) \\ & + N(k+1)P_s(k|k)\Phi_d'(k+1,k)] \\ & - \left\{K_d[k+1][M(k+1)P_s(k+1|k) + N(k+1)P_s(k|k)\Phi_d'(k+1,k)]\right\}' \\ & + K_d(k+1)\left\{M(k+1)P_s(k+1|k)M'(k+1) \right. \\ & + M(k+1)\Phi_d(k+1,k)P_s(k|k)N'(k+1) \\ & + [M(k+1)\Phi_d(k+1,k)P_s(k|k)N'(k+1)]' \\ & \left. + N(k+1)P_s(k|k)N'(k+1) + V(k+1)\right\}K_d'(k+1). \quad (3.15) \end{aligned}$$

In order to completely specify P_s , recursive relations for $R(k)$ and $S(k)$ must be found, where

$$R(k) = E[x(k)x_s'(k|k)]$$

$$\begin{aligned}
R(k) = & \Phi(k, k-1) [R(k-1) - S(k-1)] \left[I + \left\{ M(k) [\Phi(k, k-1) \right. \right. \\
& \left. \left. - \delta \Phi(k, k-1)] + N(k) \right\}' K_d'(k) \right] - \Phi(k, k-1) S(k-1) [\Phi_d(k, k-1) \\
& + K_d(k) M(k) \delta \Phi(k, k-1)]' + \Phi(k, k-1) [R'(k-1) \\
& - S(k-1)] [M(k) \Phi(k, k-1) + N(k)]' K_d'(k) . \quad (3.16)
\end{aligned}$$

Neglecting second-order effects,

$$\begin{aligned}
S(k) = & E[\hat{x}_s(k|k) \hat{x}_s'(k|k)] \\
= & \Phi_d(k, k-1) S(k-1) \Phi_d'(k, k-1) \\
& + \Phi_d(k, k-1) R'(k-1) [M(k) \Phi(k, k-1) + N(k)]' K_d'(k) \\
& - \Phi_d(k, k-1) S(k-1) [M(k) \Phi_d(k, k-1) + N(k)]' K_d'(k) \\
& + K_d'(k) \left\{ R'(k-1) [M(k) \Phi(k, k-1) + N(k)] \right. \\
& \left. - S(k-1) [M(k) \Phi_d(k, k-1) + N(k)]' \right\}' \Phi_d(k, k-1) \\
& + K_d(k) \left[[M(k) \Phi(k, k-1) + N(k)] P_s(k-1|k-1) [M(k) \Phi(k, k-1) \right. \\
& \left. + N(k)]' + [M(k) \Phi(k, k-1) + N(k)] [R(k-1) \right. \\
& \left. - S(k-1)] \delta \Phi'(k, k-1) M'(k) - \left\{ [M(k) \Phi(k, k-1) \right. \right. \\
& \left. \left. + N(k)] [R(k-1) - S(k-1)] \delta \Phi'(k, k-1) M'(k) \right\}' \right. \\
& \left. + V(k) \right] K_d'(k) . \quad (3.17)
\end{aligned}$$

In order for the above equations to be uniquely specified, initial values of $R(k)$ and $S(k)$ are required. Since $\hat{x}(0|0)$ is usually chosen as zero, $R(0)$ and $S(0)$ are null matrices.

From the implementation viewpoint, note that computation of P_s is not required in the actual operation of the filter. However, evaluation of P_s can be used to study the performance of the suboptimal filter compared to the performance of the optimal filter. In some cases, when the actual state trajectory is known in the simulation study, evaluation of P_s may not be necessary, only a comparison of the optimal estimates and the suboptimal estimates with the actual state trajectory may be sufficient to study the relative performance of the filters.

B. Performance Analysis of a Suboptimal Filter Using A Simplified Gain Computation

If the original model is used, but a suboptimal gain computation is used for some reason or another, the $P(k|k)$ given by equation (2.6) is not the covariance matrix of the a posteriori estimation error. The a priori covariance matrix of the estimation error is still given by equation (2.9) and the covariance matrix of the measurement estimation error is still given by equation (2.4).

Let $P_s(k+1|k+1)$ be the covariance matrix of the a posteriori estimation error resulting from using a suboptimal gain, $K_s(k+1) = K(k+1) + \delta K(k+1)$, in the optimal filter equations.

$P_s(k+1|k+1)$ is given by

$$\begin{aligned} P_s(k+1|k+1) &= E[\tilde{x}_s(k+1|k+1)\tilde{x}_s'(k+1|k+1)] \\ &= [I - K_s(k+1)M(k+1)]P_s(k+1|k)[I - K_s(k+1)M(k+1)]' \\ &\quad - K_s(k+1)P_s(k|k)\Phi'(k+1,k)[I - K_s(k+1)M(k+1)]' \end{aligned}$$

$$\begin{aligned}
& - \left\{ K_s(k+1) P_s(k|k) \Phi'(k+1, k) [I - K_s(k+1) M(k+1)]' \right\}' \\
& + K_s(k+1) [N(k+1) P_s(k|k) N'(k+1) + V(k+1)] K_s'(k+1). \quad (3.18)
\end{aligned}$$

Expanding (3.18) and neglecting second order effects,

$$\begin{aligned}
P_s(k+1|k+1) &= [I - K(k+1) M(k+1)] P_s(k+1|k) [I - K(k+1) M(k+1)]' \\
&+ K(k+1) P_s(k|k) \Phi'(k+1, k) [I - K(k+1) M(k+1)]' \\
&- \left\{ K(k+1) P_s(k|k) \Phi'(k+1, k) [I - K(k+1) M(k+1)]' \right\}' \\
&+ K(k+1) [N(k+1) P_s(k|k) N'(k+1) + V(k+1)] K'(k+1) \\
&- \delta K(k+1) [M(k+1) P_s(k+1|k) + \Phi(k+1, k) P_s(k|k) N'(k+1) \\
&+ N(k+1) P_s(k|k) \Phi'(k+1, k)] + \delta K(k+1) Q_s(k+1) K'(k+1) \\
&+ K(k+1) [Q_s(k+1) - M(k+1) P_s(k+1|k) M'(k+1)] \delta K'(k+1). \quad (3.19)
\end{aligned}$$

As indicated above Q_s and $P_s(k+1|k)$ are given by

$$\begin{aligned}
Q_s(k+1) &= M(k+1) P_s(k+1|k) M'(k+1) + V(k+1) + N(k+1) P_s(k|k) N'(k+1) \\
&+ M(k+1) \Phi(k+1, k) P_s(k|k) N'(k+1) \\
&+ [M(k+1) \Phi(k+1, k) P_s(k|k) N'(k+1)]' \quad (3.20)
\end{aligned}$$

and

$$P_s(k+1|k) = \Phi(k+1, k) P_s(k|k) \Phi'(k+1, k) + H(k). \quad (3.21)$$

The above equations can be used to study how suboptimal a particular filter is and how sensitive the quality of the estimates are to various methods of computing the gain matrix.

C. Definition of a Performance Index

By using the results of section A, we can obtain the perturbation in the estimation error covariance matrix for a particular design model. Likewise, using the results obtained in the last section, the perturbation in the estimation error covariance matrix can be computed from a knowledge of the perturbation in the optimal gain due to a suboptimal gain computation. Knowledge of these quantities may be useful, but their magnitude does not give a complete picture of how the overall performance of the suboptimal filter compares to the optimal filter. The information needed is contained in the value of the cost functional for the two cases.

The cost functional considered here is the mean squared estimation error. Let $J(k)$ be the value of the cost functional for the optimal filter and $J_s(k)$ be the value of the cost functional for the suboptimal filter. From the optimality of J , we know that $J_s(k) \geq J(k)$ for all k . Then

$$\begin{aligned} J(k) &= \text{tr } E \left\{ \tilde{x}(k|k) \tilde{x}'(k|k) \right\} \\ &= \text{tr } P(k|k) \\ J_s(k) &= \text{tr } E \left\{ \tilde{x}_s(k|k) \tilde{x}_s'(k|k) \right\} \\ &= \text{tr } P(k|k) + \text{tr } \delta P(k|k). \end{aligned}$$

Define a performance index $\mu(k)$ as

$$\begin{aligned}\mu(k) &= \frac{J_s(k) - J(k)}{J(k)} \\ &= \frac{\text{tr } \delta P(k|k)}{\text{tr } P(k|k)}\end{aligned}\tag{3.22}$$

We see that μ for fixed k has the properties that it is zero and a minimum when the suboptimal filter is actually optimal and is a monotonically increasing function of $\text{tr } \delta P$; i.e., the more suboptimal a filter with respect to the given cost functional, the larger the value of μ .

This performance index could be used to evaluate the relative performance of a suboptimal filter compared to the optimal one and to evaluate the relative sensitivity of the optimal filter to modeling variations.

IV. SUBOPTIMIZATION OF THE DELAYED-STATE FILTER

In this chapter, we derive a suboptimal delayed-state filter based upon a method originated by Joseph [12] and Pentecost [22]. The basic idea is to transform the original system into a collection of smaller subsystems. Optimal estimation is then performed on each of the subsystems. An estimate for the total state vector is then reconstructed from the estimates of the subsystem state vectors. This estimate is suboptimal since the subsystem estimators are unable to make full use of the information available regarding the total state, because such information is distributed among the subsystems. In practice, the estimates of the subsystem state vectors are not needed to reconstruct the total system state vector estimate; only the subsystem gain matrices are needed.

For convenience, we repeat the total system model equations as given in chapter II:

$$x(k+1) = \Phi(k+1, k)x(k) + u(k) \quad (4.1)$$

$$y(k+1) = M(k+1)x(k+1) + N(k+1)x(k) + v(k+1).$$

The previous assumptions about this model made in Chapter II are assumed here also.

Let n_j be the dimension of the j^{th} subsystem, where $\sum_{j=1}^r n_j \geq n$ and r is the number of subsystems. Let m_j be the dimension of the j^{th} subsystem measurement vector, $\sum_{j=1}^r m_j \geq m$. Define the j^{th} subsystem state

vector ξ_j and the j^{th} subsystem measurement vector η_j by

$$\begin{aligned}\xi_j(k) &= C_j x(k) \\ \eta_j(k) &= D_j y(k)\end{aligned}\tag{4.2}$$

where the linear transformations C_j and D_j are $n_j \times n$ and $m_j \times m$ respectively and have maximal rank. We choose C_j and D_j to be time-invariant to avoid any more complexity in the filter than necessary.

At this point, we could, by sheer brute-force, proceed to design the optimal subsystem estimator as in the design of the optimal estimator for the total system. However, we will take a simpler route by manipulating the subsystem equations into the form of (4.1) and then make use of the results already available in the appendix.

From (4.2), we approximate x and y by

$$\begin{aligned}x(k) &= C_j^+ \xi_j(k) \\ y(k) &= D_j^+ \eta_j(k)\end{aligned}\tag{4.3}$$

where the superscript plus denotes the pseudo-inverse; i.e.,

$$C_j^+ = C_j' (C_j C_j')^{-1}.$$

Premultiplying the first equation of (4.1) by C_j and the second equation by D_j and substituting in (4.2) and (4.3), we obtain

$$\begin{aligned}\xi_j(k+1) &= C_j \Phi(k+1, k) C_j^+ \xi_j(k) + C_j u(k) \\ \eta_j(k+1) &= D_j M(k+1) C_j^+ \xi_j(k+1) + D_j N(k+1) C_j^+ \xi_j(k) + D_j v(k+1)\end{aligned}\tag{4.4}$$

Denoting Φ_j , M_j , N_j , u_j , and v_j by

$$\begin{aligned}
 \Phi_j(k+1, k) &= C_j \Phi(k+1, k) C_j^+ \\
 M_j(k+1) &= D_j M(k+1) C_j^+ \\
 N_j(k+1) &= D_j N(k+1) C_j^+ \\
 u_j(k) &= C_j u(k) \\
 v_j(k+1) &= D_j v(k+1)
 \end{aligned} \tag{4.5}$$

we see that the subsystem equations are of the same form as (4.1).

$$\begin{aligned}
 \xi_j(k+1) &= \Phi_j(k+1, k) \xi_j(k) + u_j(k) \\
 \eta_j(k+1) &= M_j(k+1) \xi_j(k+1) + N_j(k+1) \xi_j(k) + v_j(k+1).
 \end{aligned} \tag{4.6}$$

Now that the subsystem equations have been put into the proper form, it is a routine matter to apply the results of Brown and Hartman [7] to each subsystem. For the j^{th} subsystem, the optimal delayed-state filter equations are

$$\begin{aligned}
 Q_j(k) &= M_j(k) P_j(k|k-1) M_j'(k) + V_j(k) + N_j(k) P_j(k-1|k-1) N_j'(k) \\
 &\quad + M_j(k) \Phi_j(k, k-1) P_j(k-1|k-1) N_j'(k) \\
 &\quad + [M_j(k) \Phi_j(k, k-1) P_j(k-1|k-1) N_j'(k)]'
 \end{aligned} \tag{4.7}$$

$$K_j(k) = [P_j(k|k-1) M_j'(k) + \Phi_j(k, k-1) P_j(k-1|k-1) N_j'(k)] Q_j^{-1}(k) \tag{4.8}$$

$$P_j(k|k) = P_j(k|k-1) - K_j(k) Q_j(k) K_j'(k) \tag{4.9}$$

$$\hat{\xi}_j(k|k) = \hat{\xi}_j(k|k-1) + K_j(k) [\eta_j(k) - M_j(k) \hat{\xi}_j(k|k-1) - N_j(k) \hat{\xi}_j(k-1|k-1)] \quad (4.10)$$

$$\hat{\xi}_j(k+1|k) = \Phi_j(k+1, k) \hat{\xi}_j(k|k) \quad (4.11)$$

$$P_j(k+1|k) = \Phi_j(k+1, k) P_j(k|k) \Phi_j(k+1, k) + H_j(k) \quad (4.12)$$

where

$$V_j(k) = E \{ v_j(k) v_j'(k) \} = D_j V(k) D_j'$$

and

$$H_j(k) = E \{ u_j(k) u_j'(k) \} = C_j H(k) C_j'$$

We assume that the estimate of the total system state vector can be reconstructed from the estimates of the subsystem state vectors by the relations

$$\hat{x}(k|k-1) = \sum_{j=1}^r F_j \hat{\xi}_j(k|k-1) \quad (4.13)$$

$$\hat{x}(k|k) = \sum_{j=1}^r F_j \hat{\xi}_j(k|k) \quad (4.14)$$

where the F_j and C_j must satisfy

$$\sum_{j=1}^r F_j C_j = I \quad (4.15)$$

in order that the total state vector is reconstructed from the subsystem

state vectors.

From (4.14) and (4.10), we have

$$\begin{aligned}
 \hat{x}(k|k) &= \sum_{j=1}^r F_j \hat{\xi}_j(k|k-1) + \sum_{j=1}^r F_j K_j(k) D_j [y(k) - M(k) \hat{x}(k|k-1) \\
 &\quad - N(k) \hat{x}(k-1|k-1)] \\
 &= \hat{x}(k|k-1) + \sum_{j=1}^r F_j K_j(k) D_j [y(k) \\
 &\quad - M(k) \hat{x}(k|k-1) - N(k) \hat{x}(k-1|k-1)] .
 \end{aligned} \tag{4.16}$$

A comparison of (4.16) and (2.15) shows that the partitioned-state estimation method implies that

$$K(k) = \sum_{j=1}^r F_j K_j(k) D_j. \tag{4.17}$$

It is now obvious from equations (4.16) and (4.17) that estimates of the subsystem state vectors are not needed in order to estimate the total system state vector. With the estimator now in the form of (4.16), the constraint (4.15) is no longer needed.

Note that $\hat{\xi}_j(0|0) = D_j \hat{x}(0|0)$ and

$$\begin{aligned}
 P_j(0|0) &= E \left\{ [\xi_j(0) - \hat{\xi}_j(0|0)] [\xi_j(0) - \hat{\xi}_j(0|0)]' \right\} \\
 &= C_j P(0|0) C_j'.
 \end{aligned} \tag{4.18}$$

Thus, the suboptimal filter requires the same initial conditions as the optimal filter. A schematic diagram of the suboptimal filter is given

in Fig. 3.

Summarizing the suboptimal filtering equations, for $j=1, \dots, r$,

$$\hat{x}(k|k-1) = \Phi(k, k-1) \hat{x}(k-1|k-1) \quad (4.19)$$

$$\Phi_j(k, k-1) = C_j \Phi(k, k-1) C_j^+ \quad (4.20)$$

$$P_j(k|k-1) = \Phi_j(k, k-1) P_j(k-1|k-1) \Phi_j'(k, k-1) + H_j(k-1) \quad (4.21)$$

$$\begin{aligned} Q_j(k) = & M_j(k) P_j(k|k-1) M_j'(k) + V_j(k) + N_j(k) P_j(k-1|k-1) N_j'(k) \\ & + M_j(k) \Phi_j(k, k-1) P_j(k-1|k-1) N_j'(k) \\ & + [M_j(k) \Phi_j(k, k-1) P_j(k-1|k-1) N_j'(k)]' \end{aligned} \quad (4.22)$$

$$K_j(k) = [P_j(k|k-1) M_j'(k) + \Phi_j(k, k-1) P_j(k-1|k-1) N_j'(k)] Q_j^{-1}(k) \quad (4.23)$$

$$P_j(k|k) = P_j(k|k-1) - K_j(k) Q_j(k) K_j'(k) \quad (4.24)$$

$$\begin{aligned} \hat{x}(k|k) = & \hat{x}(k|k-1) + K(k) [y(k) - M(k) \hat{x}(k|k-1) \\ & - N(k) \hat{x}(k-1|k-1)] \end{aligned} \quad (4.25)$$

where $K(k)$ is given by equation (4.17).

Equation (4.24) is true only if the gain used is the optimal gain for the j^{th} subsystem and is computed exactly. If K_j is not computed exactly (truncation errors may occur), it is not guaranteed that $P_j(k|k)$ will remain positive definite. To avoid this problem, it is desirable to have an alternate way of computing $P_j(k|k)$ which will assure positive definiteness. The required form is

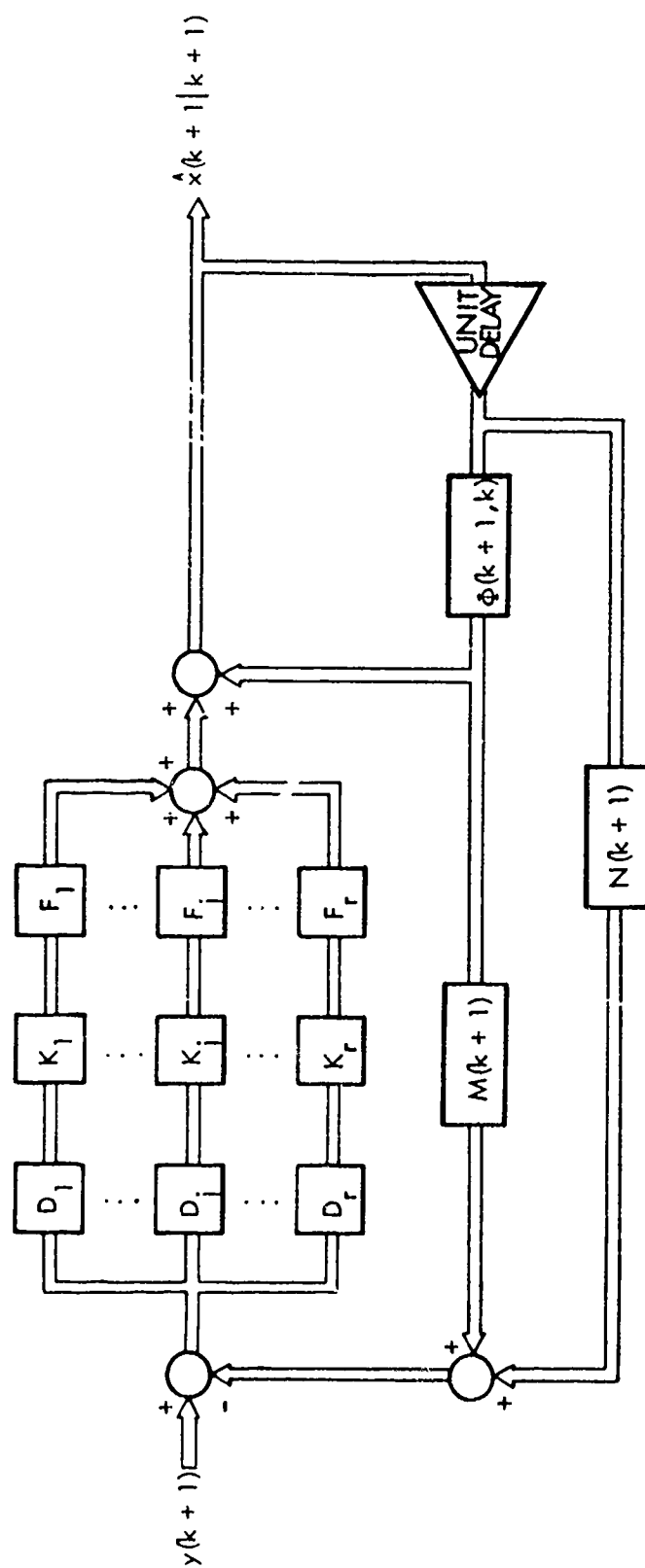


Fig. 3. Structure of the partitioned-state suboptimal filter

$$\begin{aligned}
P_j(k|k) = & \left\{ \Phi_j(k, k-1) - K_j(k)[M_j(k)\Phi_j(k, k-1) + N_j(k)] \right\} P_j(k-1|k-1) \\
& \left\{ \Phi_j(k, k-1) - K_j(k)[M_j(k)\Phi_j(k, k-1) + N_j(k)] \right\}' \\
& + K_j(k)V_j(k)K_j'(k) + H(k-1). \quad (4.26)
\end{aligned}$$

By using the above equation and equation (4.21), the a priori subsystem estimation error, covariance matrix can be eliminated from equations (4.22) and (4.23) for Q_j and K_j respectively. Q_j is then given by

$$\begin{aligned}
Q_j(k) = & M_j(k)\Phi_j(k, k-1)P_j(k-1|k-1)\Phi_j'(k, k-1)M_j'(k) \\
& + M_j(k)H_j(k-1)M_j'(k) + V_j(k) + N_j(k)P_j(k-1|k-1)N_j'(k) \\
& + M_j(k)\Phi_j(k, k-1)P_j(k-1|k-1)N_j'(k) \\
& + [M_j(k)\Phi_j(k, k-1)P_j(k-1|k-1)N_j'(k)]' \quad (4.27)
\end{aligned}$$

and K_j is given by

$$\begin{aligned}
K_j(k) = & \left\{ \Phi_j(k, k-1)P_j(k-1|k-1)[M_j(k)\Phi_j(k, k-1) + N_j(k)]' \right. \\
& \left. + H_j(k-1)M_j'(k) \right\} Q_j^{-1}(k). \quad (4.28)
\end{aligned}$$

Notice that the computations in equations (4.21)-(4.24) involve matrices of order less than $m \times m$ and $n \times n$. The partitioning of the system into subsystems is not unique. The optimum partitioning for a particular problem at this time must be found by trial and error and physical intuition. If the subsystem state vectors are not cross-coupled with any other of the subsystem state vectors, either through the plant or measurement equations or through the noise processes, then this

scheme would be optimal, as indicated by Aoki [1] and Pentecost [22]. At the present time, there is no optimal method for choosing the F_j matrices. Usually, they are taken to be C_j^+ .

As an aid in evaluating the reduction in the amount of required computation achieved by the use of the above results, it is useful to compute the number of multiplication and addition operations required in one step by the filters.

Assuming the inverse is computed using the gaussian elimination method, the number of multiplication operations M_o and the number of addition operations A_o required in one step by the optimal filter equations given in chapter II are given by

$$M_o = 2n^3 + n^2(1+7m) + nm(5+4m) + m^3 \quad (4.29)$$

$$A_o = 2n^3 + 7n^2m - 3nm + 3nm^2 + m(m^2+2) - n. \quad (4.30)$$

The number of multiplication operations M_s and the number of addition operations A_s required in one step by the suboptimal filter equations (4.17), (4.19)-(4.25) are given by

$$M_s = n^2 + 5nm + \sum_{j=1}^r [2n_j^3 + 6n_jm_j^2 + 8n_j^2m_j + n_jnm + m_j^3] \quad (4.31)$$

$$A_s = m(3n+1) + \sum_{j=1}^r [2n_j^3 + 8n_j^2m_j + n_jm_j(4m_j-7) - n_j^2 + n_jm(m_j+n-1) + m_j + m_j^3]. \quad (4.32)$$

As an example, consider a sixteen-dimensional system with a scalar measurement. Three subsystems are chosen, two are three-dimensional and one is twelve-dimensional. The scalar measurement is used in all three subsystems. Then

$$\begin{array}{ll} M_o = 10385 & M_s = 5615 \\ A_o = 9971 & A_s = 5011 \end{array}$$

As a consequence of using the partitioned suboptimal filter, a 56% reduction in the number of multiplication operations is achieved and a 50% reduction in the number of addition operations required in one step. The substantial reduction in the multiplication operations is particularly significant since multiplication operations are much more costly in terms of computer execution time than addition operations.

The question of how suboptimal this method is may be investigated in the following manner. Let K be the gain matrix of the optimal delayed-state filter. Then the perturbation of this optimal gain incurred by using the above partitioning scheme is

$$\delta K(k) = \sum_{j=1}^r F_j K_j(k) D_j - K(k). \quad (4.33)$$

δK thus calculated can then be used in the performance analysis equations of chapter III to calculate μ , the performance index, μ is a relative measure of how suboptimal a particular partitioning is for a specific system and can be used to evaluate several partitionings to determine which one is the closest to being optimal (least suboptimal).

V. AN EXAMPLE: NNSS INTEGRATED INERTIAL/DOPPLER-
SATELLITE NAVIGATION SYSTEM

In order to illustrate how the preceding results may be applied, we consider an inertial navigation system augmented by the Navy Navigation Satellite System (NNSS), also referred to as the Transit System. In the next three sections, we will present the model of the system and derive expressions for the noise covariance matrices. After some general comments on the method used to simulate the system, we will present the results of two performance analyses for modeling changes and the results on the performance of a partitioned-state suboptimal filter.

For our example, we consider a terrestrial (low-altitude) vehicle using a strapped-down inertial system, with the vertical channel being implemented by other than inertial means. A geocentric latitude-longitude coordinate system is used.

Such an inertial navigation system is capable of providing global navigational information, but suffers from long term "drift" errors. To compensate for this "drift," another source of information may be used to periodically correct the inertial system. That is the role of the NNSS here. The role of the Kalman filter is to integrate the two systems in an optimal fashion. A schematic representation of the integrated system is shown in Fig. 4.

Periodically the vehicle receives a message signal from one of the satellites. The message signal gives the satellite's position and other

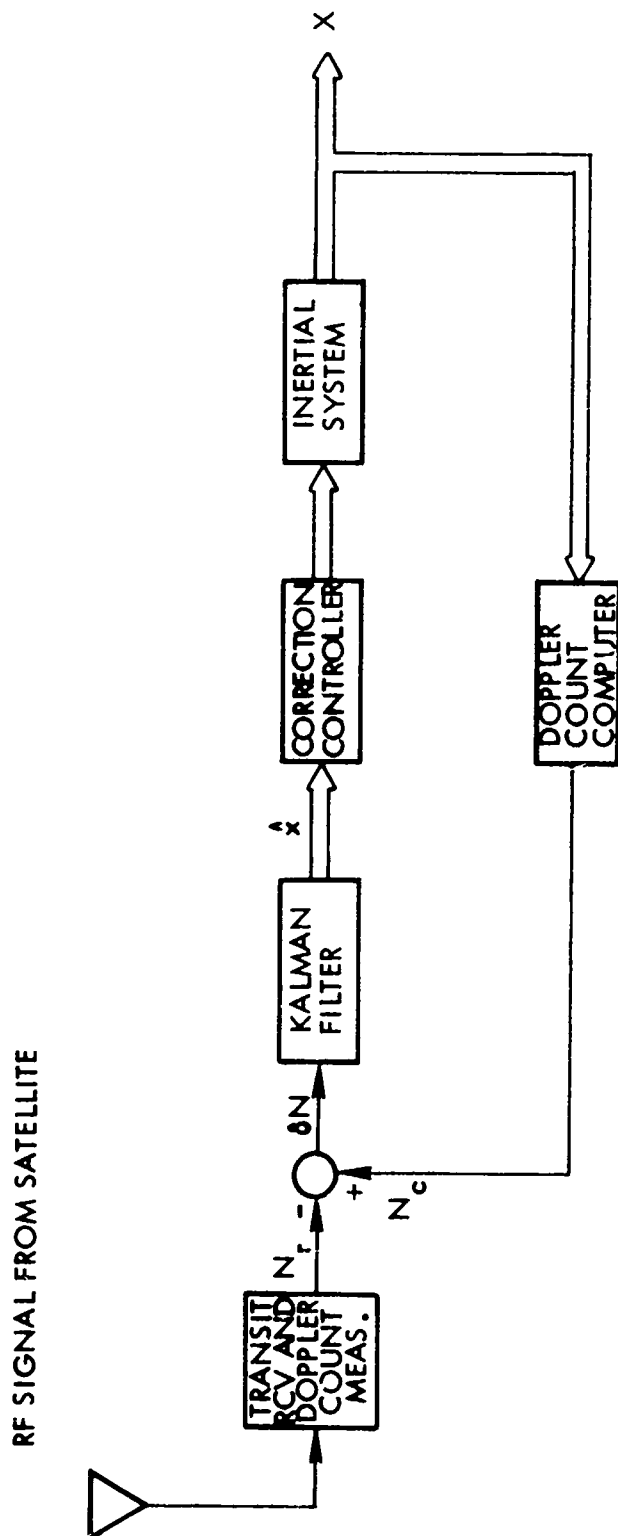


Fig 4. NNSS integrated inertial/doppler-satellite navigation system

miscellaneous information. Due to the relative velocity between the satellite and the vehicle, a doppler shift in the signal frequency occurs. Using the satellite's position and a measurement of the accumulated doppler count over a time interval (approximately twenty seconds), information about the inertial system's errors can be inferred. Since the measurement process is noisy and the errors in the inertial system are inherently random, a Kalman filter can be used to obtain a better estimate of the inertial system errors than could be obtained by using the raw measurement.

As we shall see in Section B, a delayed-state filter is required because the measurement vector (a scalar in this case) depends upon the number of doppler counts accumulated over the interval between sample times. The fact that the measurement vector is a scalar makes this system particularly attractive from a computational point of view since the Q matrix is a scalar and taking its inverse is a trivial operation. On the other hand, a delayed-state filter, which places greater demands on the computational ability of the on-board computer than the usual Kalman filter, is required.

A. The Plant Model

The error model for the inertial system may be described by two basic equations, Pitman [23]. The two equations are

$$\dot{\mathbf{q}} + \boldsymbol{\omega} \times \mathbf{q} = \boldsymbol{\epsilon} \quad (5.1)$$

$$\delta \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \times \dot{\delta \mathbf{R}} + \dot{\boldsymbol{\omega}} \times \delta \mathbf{R} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \delta \mathbf{R} = \delta \mathbf{a} - \boldsymbol{\psi} \times \mathbf{a} - \omega_o^2 \delta \mathbf{R}_{\text{tan}} \quad (5.2)$$

where all the above variables are three-dimensional vectors, the cross \times indicates the vector cross product, and

$$\boldsymbol{\psi} = \boldsymbol{\phi} - \delta \boldsymbol{\theta},$$

$\boldsymbol{\phi}$ = platform coordinate frame error vector,

$\delta \boldsymbol{\theta}$ = computer coordinate frame error vector,

$\boldsymbol{\omega}$ = platform angular rate vector with respect to an inertial frame of reference,

$\boldsymbol{\epsilon}$ = gyro "drift" rate (bias) error vector,

$\delta \mathbf{R}$ = radial position error vector,

$\delta \mathbf{a}$ = accelerometer "bias" error vector,

\mathbf{a} = sensed "acceleration" vector (including both inertial and mass attraction acceleration),

$\delta \mathbf{R}_{\text{tan}}$ = tangential component of \mathbf{R} to the earth,

$\omega_o^2 = g_m/R$, g_m is the mass attraction acceleration and R is the nominal magnitude of the earth's radius vector plus the nominal altitude of the vehicle.

Equation (5.1) describes the "twenty-four hour dynamics" in terms of the difference between the angular error in the orientation of the platform frame and the computer frame. Equation (5.2) describes the position error propagation, the "eighty-four minute" or "Shuler" dynamics. The third component equation in (5.2) will not be used since that channel is to be implemented by other means, such as by an altimeter.

Since the system is a strapped-down system, the gyro drift errors and accelerometer biases given in the platform frame must be transformed through a continuously updated direction cosine matrix into the computer frame to be used in equations (5.1) and (5.2).

The random process driving functions in the plant equations (5.1) and (5.2) are the gyro drift rate and the accelerometer bias. To fit the form required by the Kalman filter, these processes must be gaussian white noise with zero mean. By physical observation, this is obviously not true. The standard method for circumventing this difficulty will be employed. We consider the components of ϵ and δa to be correlated noise processes driven by gaussian white noise with zero mean.

More specifically, we assume that the components of ϵ and δa are first order Markov processes of the form

$$\dot{x} + \beta x = \sqrt{2\sigma^2\beta} f(t) \quad (5.3)$$

where $f(t)$ is unity white noise. Augmenting the state with six additional states due to ϵ and δa , the plant model is now in the form required by the Kalman filter.

In addition, three more state variables are needed to complete the model. One is needed for the vertical position error, one for the vertical velocity error, and one for the doppler count bias error.

The vertical velocity error and the count bias error will be modeled by an equation of the form of (5.3). The model is one which can be made to fit a variety of situations and implementations by adjustment of the

parameters σ and β . For very small β , the process approaches a true bias and for very large β the process approaches white noise. The amplitude is adjusted by σ .

The vertical position error is taken to be the integral of the vertical velocity error and is thus a nonstationary process.

The coordinate frames to be used are defined as follows:

- X, Y, Z = earth-fixed frame with X through the north pole
and Z through the intersection of the equator and
the Greenwich meridian,
- x, y, z = geocentric navigation frame with x north, y west,
and z up
- x', y', z' = body-mounted instrument-cluster reference frame
with x' nose, y' left wing, and z' through the
roof.

The relationships of the three frames are shown in Fig. 5.

For convenience, we replace δR_x and δR_y in equation (14) by their angular equivalents in terms of $\delta\theta_x$ and $\delta\theta_y$,

$$\delta R_x = R\delta\theta_y$$

$$\delta R_y = -R\delta\theta_x,$$

and define state variables as follows:

$$x_1 = \psi_x$$

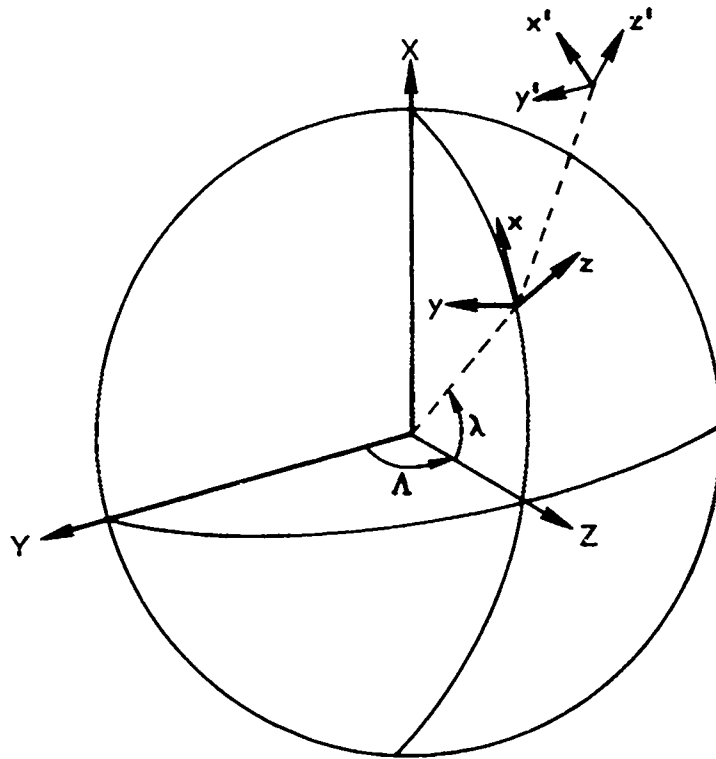


Fig. 5. Relationship between the XYZ, xyz, and $x'y'z'$ coordinate frames

$$x_2 = \dot{\psi}_y$$

$$x_3 = \dot{\psi}_z$$

$$x_4 = \delta \dot{\theta}_x$$

$$x_5 = \delta \dot{\theta}_x / \omega_0$$

$$x_6 = \delta \dot{\theta}_y$$

$$x_7 = \delta \dot{\theta}_y / \omega_0$$

$$x_8 = \delta R_z / R$$

$$x_9 = \epsilon_{x'} / \omega_o$$

$$x_{10} = \epsilon_{y'} / \omega_o$$

$$x_{11} = \epsilon_{z'} / \omega_o$$

$$x_{12} = a_{x'} / R\omega_o^2$$

$$x_{13} = \delta a_{y'} / R\omega_o^2$$

$$x_{14} = \delta a_{z'} / R\omega_o^2$$

$$x_{15} = \delta \dot{R}_z / R\omega_o^2$$

$$x_{16} = \delta N$$

where δN is the doppler count bias error. Note that the state variables have been chosen to be dimensionless quantities.

After a great deal of algebraic manipulation, we obtain the following nonzero elements of the A matrix (Brown [5]):

$$a_{1,2} = \omega_z$$

$$a_{1,3} = -\omega_y$$

$$a_{1,9} = \omega_o c_{xx'}$$

$$a_{1,10} = \omega_o c_{xy'}$$

$$a_{1,11} = \omega_o C_{xz},$$

$$a_{2,1} = -\omega_z$$

$$a_{2,3} = \omega_x$$

$$a_{2,9} = \omega_o C_{xz},$$

$$a_{2,10} = \omega_o C_{yy},$$

$$a_{2,11} = \omega_o C_{yz},$$

$$a_{3,1} = \omega_y$$

$$a_{3,2} = -\omega_x$$

$$a_{3,9} = \omega_o C_{zx},$$

$$a_{3,10} = \omega_o C_{zy},$$

$$a_{3,11} = \omega_o C_{zz},$$

$$a_{4,5} = \omega_o$$

$$a_{5,1} = -\frac{\Delta V_z}{\Delta t} \frac{1}{R\omega_o} + \frac{1}{\omega_o} (\omega_x^2 + \omega_y^2 - \omega_o^2)$$

$$a_{5,3} = \frac{\dot{\omega_y R}}{R\omega_o} + \frac{1}{R\omega_o} \frac{\Delta(\omega_y R)}{\Delta t} + \frac{1}{\omega_o} \omega_x \omega_z$$

$$a_{5,4} = - \frac{\Delta V}{\Delta t} \frac{1}{R\omega_o} + \frac{1}{\omega_o} (\omega_x^2 + \omega_z^2 - \omega_o^2)$$

$$a_{5,5} = \frac{-2\Delta R}{\Delta t R}$$

$$a_{5,6} = \frac{\omega_z \dot{R}}{R\omega_o} + \frac{\Delta(\omega_z R)}{R\omega_o \Delta t} + \frac{\omega_x \omega_y}{\omega_o}$$

$$a_{5,7} = 2\omega_z$$

$$a_{5,8} = \frac{\omega_z \omega_y}{\omega_o} - \frac{\Delta\omega_x}{\omega_o \Delta t}$$

$$a_{5,12} = - \omega_o C_{yx'}$$

$$a_{5,13} = - \omega_o C_{yy'}$$

$$a_{5,14} = - \omega_o C_{yz'}$$

$$a_{5,15} = - 2\omega_x$$

$$a_{6,7} = \omega_o$$

$$a_{7,2} = - \frac{\Delta V}{R\omega_o \Delta t} + \frac{1}{\omega_o} (\omega_x^2 + \omega_y^2 - \omega_o^2)$$

$$a_{7,3} = \frac{\omega_x \dot{R}}{R\omega_o} + \frac{\Delta(\omega_x R)}{R\omega_o \Delta t} - \frac{\omega_y \omega_z}{\omega_o}$$

$$a_{7,4} = - \frac{\dot{\omega_z R}}{R\omega_o} - \frac{\Delta(\omega_z R)}{R\omega_o \Delta t} + \frac{\omega_x \omega_y}{\omega_o}$$

$$a_{7,5} = - 2\omega_z$$

$$a_{7,6} = - \frac{\Delta V_z}{R\omega_o \Delta t} + \frac{1}{\omega_o} (\omega_y^2 + \omega_z^2 - \omega_o^2)$$

$$a_{7,7} = - \frac{2\Delta R}{R\Delta t}$$

$$a_{7,8} = - \frac{\Delta\omega_y}{\omega_o \Delta t} - \frac{\omega_x \omega_z}{\omega_o}$$

$$a_{7,12} = \omega_o C_{xx},$$

$$a_{7,13} = \omega_o C_{xy},$$

$$a_{7,14} = \omega_o C_{xz},$$

$$a_{7,15} = - 2\omega_y$$

$$a_{8,15} = \omega_o$$

$$a_{9,9} = - \beta_2$$

$$a_{10,10} = - \beta_3$$

$$a_{11,11} = -\beta_4$$

$$a_{12,12} = -\beta_5$$

$$a_{13,13} = -\beta_6$$

$$a_{14,14} = -\beta_7$$

$$a_{15,15} = -\beta_1$$

$$a_{16,16} = -\beta_8$$

where C denotes the direction cosine between the subscripted coordinate axes, Δ denotes the incremental change in the indicated quantities over the time interval $\Delta t = t_{k+1} - t_k$, and average values over the Δt interval are implied for the other quantities. Assuming a small Δt compared to the time constants of the dynamics, average values are used so that an exponential routine can be used to compute the transition matrix.

Notice that the A matrix is in the general partitioned form:

$$A = \left[\begin{array}{c|c} \alpha_{11} & \alpha_{12} \\ (8 \times 8) & (8 \times 8) \\ \hline \emptyset & \alpha_{22} \\ (8 \times 8) & (8 \times 8) \\ & \text{diagonal} \end{array} \right] .$$

The state variables were ordered so as to simplify the computation of the transition matrix.

B. The Measurement Model

In this section, we derive the measurement model for the integrated NNSS. The approach is the same as in Brown [5] and Brown and Hagerman [6].

The equation for the ideal count observed at the receiver for the time interval (t_{k-1}, t_k) is given in Stansell [30],

$$N(k) = \Delta F \Delta T + \frac{1}{\lambda_G} [\rho(k) - \rho(k-1)] , \quad (5.4)$$

where

ΔF = fixed offset in transmitter and local oscillator frequencies (32 kHz),

ΔT = time interval between timing marks referred to the satellite time base,

λ_G = wavelength of the local oscillator,

ρ = range from the observer to the satellite.

A number of sources of error are present in the count measurement process. First, oscillators are not so stable that the local oscillator offset can be maintained at 32 kHz indefinitely, so some unknown bias effect will be present. Secondly, refraction due to the atmosphere affects the rf signal from the satellite. This is particularly true when

the satellite passes low over the horizon. Lastly, jitter in the counting electronics introduces additional random error.

These sources of error in the count measurement will be considered as the sum of a correlated process and an uncorrelated process. The correlated part we model as a first order Markov process and designate as state variable x_{16} . The uncorrelated part we assume to be the gaussian white noise process in our measurement model. Thus, the measured count is given by

$$N_m(k) = \Delta F \Delta T + \frac{1}{\lambda_G} [\rho(k) - \rho(k-1)] - \delta N(k) - v(k). \quad (5.5)$$

If the inertial system were without error, the computed count would be given by equation (5.4). But since the inertial system is in error, the range computed will be in error by some amount. Thus, the computed count is given by

$$N_c(k) = \Delta F \Delta T + \frac{1}{\lambda_G} [\rho(k) + \delta \rho(k) - \rho(k-1) - \delta \rho(k-1)] \quad (5.6)$$

where $\delta \rho$ is the error in the range due to the position error in the inertial system. We assume that the position of the satellite is known perfectly.

Taking the difference between N_c and N_m , we obtain

$$y(k) = N_c(k) - N_m(k) = \frac{1}{\lambda_G} [\delta \rho(k) - \delta \rho(k-1)] + \delta N(k) + v(k). \quad (5.7)$$

To obtain the form required by the delayed-state filter, we must examine the $\delta \rho$ terms. The expression for $\delta \rho$ is given in Brown and Hagerman [7] as

$$\delta \rho = \frac{1}{\rho} \left[(R - R_s C_{zz_s}) \delta R_z + (RR_s C_{yz_s}) \delta \theta_x - (RR_s C_{xz_s}) \delta \theta_y \right] \quad (5.8)$$

where R is the radial distance from the center of the earth to the vehicle, R_s is the radial distance from the center of the earth to the satellite, and C_{xz_s} , C_{yz_s} , and C_{zz_s} are the direction cosines between the navigation axes and the satellite z_s axis which coincides with the radial vector between the center of the earth and the satellite.

The range is computed from

$$\rho = (R^2 + R_s^2 - 2 RR_s C_{zz_s})^{\frac{1}{2}}. \quad (5.9)$$

Noting that $\delta \theta_x = x_4$, $\delta \theta_y = x_6$, $\delta R_z/R = x_8$, and $\delta N = x_{16}$, the measurement model now fits the required form,

$$y(k) = M(k)x(k) + N(k)x(k-1) + v(k). \quad (5.10)$$

where

$$M(k) = \begin{bmatrix} 0 & 0 & 0 & b(k) & 0 & c(k) & 0 & Ra(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$N(k) = \begin{bmatrix} 0 & 0 & 0 & -b(k-1) & 0 & c(k-1) & 0 & -Ra(k-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a(k) = \frac{1}{G^2} [R - R_s C_{zz_s}],$$

$$b(k) = \frac{1}{G^2} [RR_s C_{yz_s}],$$

and

$$c(k) = -\frac{1}{\lambda G^0} [RR_s \ C_{xz_s}] .$$

The model for the system is now complete. We turn now to the derivation of the noise covariance matrices to be used in the filter.

C. The Noise Covariance Matrices

The derivation of the nonzero terms in the H matrix is a straight forward exercise in classical random process theory. The value of the V matrix (a scalar) is based on physical intuition and simulation results, Brown [5].

By inspection, the first three diagonal terms are white noise constants due to gyro drift.

$$h_{1,1} = u_1^2$$

$$h_{2,2} = u_2^2$$

$$h_{3,3} = u_3^2$$

Similarly, the fifth and seventh diagonal terms are white noise constants due to accelerometer bias.

$$h_{5,5} = u_5^2$$

$$h_{7,7} = u_7^2$$

The state equation for the first gyro drift component, x_9 , is

$$\dot{x}_9(t) = -\beta_2 x_9(t) + \sqrt{2\sigma_2^2 \beta_2} f(t) . \quad (5.11)$$

Solving equation (5.11) over the time interval (t_k, t_{k+1}) ,

$$x_9(k+1) = \exp(-\beta_2 \Delta t) x_9(k) + \int_0^{\Delta t} \sqrt{2\sigma_2^2 \beta_2} \exp(-\beta_2 \tau) f(t-\tau) d\tau \quad (5.12)$$

where $\Delta t = t_{k+1} - t_k$. The $h_{9,9}(k)$ term is the covariance of the response of $x_9(k+1)$ due to $f(t)$ between t_k and t_{k+1} ; i.e.,

$$\begin{aligned} h_{9,9}(k) &= E \left[\int_0^{\Delta t} \sqrt{2\sigma_2^2 \beta_2} \exp(-\beta_2 \tau) f(t-\tau) d\tau \int_0^{\Delta t} \sqrt{2\sigma_2^2 \beta_2} \exp(-\beta_2 s) f(t-s) ds \right] \\ &= 2\sigma_2^2 \beta_2 \int_0^{\Delta t} \int_0^{\Delta t} \exp[-\beta_2(\tau+s)] E[f(t-\tau) f(t-s)] d\tau ds \\ &= 2\sigma_2^2 \beta_2 \int_0^{\Delta t} \int_0^{\Delta t} \exp[-\beta_2(\tau+s)] \delta(\tau-s) d\tau ds \\ &= \sigma_2^2 [1 - \exp(-2\beta_2 \Delta t)] \end{aligned}$$

where δ is used here to denote the Dirac delta function.

Similarly:

$$h_{10,10}(k) = \sigma_3^2 [1 - \exp(-2 \beta_3 \Delta t)]$$

$$h_{11,11}(k) = \sigma_4^2 [1 - \exp(-2 \beta_4 \Delta t)]$$

$$h_{12,12}(k) = \sigma_5^2 [1 - \exp(-2 \beta_5 \Delta t)]$$

$$h_{13,13}(k) = \sigma_6^2 [1 - \exp(-2 \beta_6 \Delta t)]$$

$$h_{14,14}(k) = \sigma_7^2 [1 - \exp(-2 \beta_7 \Delta t)]$$

$$h_{15,15}(k) = \sigma_1^2 [1 - \exp(-2 \beta_1 \Delta t)]$$

$$h_{16,16}(k) = \sigma_8^2 [1 - \exp(-2 \beta_8 \Delta t)] .$$

The state variable x_8 is merely the integral of x_{15} , thus

$$\begin{aligned} h_{8,8}(k) &= \frac{2\omega_c^2 \sigma_1^2}{\beta_1} \int_0^{\Delta t} \int_0^{\Delta t} [1 - \exp(-\beta_1 \tau)][1 - \exp(-\beta_1 s)] \delta(\tau-s) \, d\tau ds \\ &= \frac{2\omega_c^2 \sigma_1^2}{\beta_1^2} \left\{ \beta_1 \Delta t - 2[1 - \exp(-\beta_1 \Delta t)] + \frac{1}{2}[1 - \exp(-2\beta_1 \Delta t)] \right\}. \end{aligned}$$

The only cross-covariance terms which are present are $h_{8,15}$ and $h_{15,8}$ which are equal.

$$\begin{aligned}
 h_{8,15}(k) &= h_{15,8}(k) \\
 &= \int_0^{\Delta t} \int_0^{\Delta t} 2\sigma_1^2 \omega_o [1 - \exp(-\beta_1 \tau)] \exp(-\beta_1 s) \delta(\tau-s) d\tau ds \\
 &= \frac{2\sigma_1^2 \omega_o}{\beta_1} \left\{ [1 - \exp(-\beta_1 \Delta t)] - \frac{1}{2} [1 - \exp(-2\beta_1 \Delta t)] \right\} .
 \end{aligned}$$

Of course, values of the σ 's and the β 's must be determined. In our simulations, we shall assume values which we presume correctly characterize the random processes. We will base our choice of values on simulations of this system.

D. General Comments on the Simulation Study

A few comments are in order concerning the values of the statistical parameters used in our simulations and how they were obtained.

The value of $P(0|0)$ is based upon the discussion in Brown [5] on establishing initial conditions for the optimal filter for the system considered here. The actual numerical values are the result of a trial and error simulation study of the performance of the optimal filter using actual flight test data. The optimal filter performed "best" when the chosen parameters were used. Upon that basis, the selected values are

assumed to correctly characterize the random processes involved for the purposes of this study.

The values in the initial covariance matrix are chosen in the xyz navigation coordinate system and are then referred, via a linear transformation into the body-mounted coordinate system, in which the actual initial alignment of the system takes place.

In the xyz coordinate system, the initial estimation error covariance matrix is a diagonal matrix $P^*(0|0)$ with

$$P_{1,1}^* = 800 \widehat{\text{sec}}^2$$

$$P_{9,9}^* = 9 \times 10^{-4} / \omega_o^2 \left(\frac{\widehat{\text{sec}}}{\text{sec}} \right)^2$$

$$P_{2,2}^* = P_{1,1}^*$$

$$P_{10,10}^* = P_{9,9}^*$$

$$P_{3,3}^* = 100 \widehat{\text{min}}^2$$

$$P_{11,11}^* = P_{9,9}^*$$

$$P_{4,4}^* = 100/R^2 \text{ (R in feet)}$$

$$P_{12,12}^* = 400 \widehat{\text{sec}}^2$$

$$P_{5,5}^* = 0.01/\omega_o^2 R^2 (\text{sec}^{-1})^2$$

$$P_{13,13}^* = P_{12,12}^*$$

$$P_{6,6}^* = P_{4,4}^*$$

$$P_{14,14}^* = P_{12,12}^*$$

$$P_{7,7}^* = P_{5,5}^*$$

$$P_{15,15}^* = 0.04/\omega_o^2 R^2 (\text{sec}^{-1})^2$$

$$P_{8,8}^* = 40000/R^2$$

$$P_{16,16}^* = 10^6$$

The linear transformation relating the state in the xyz coordinate system to the state in the x'y'z' coordinate system is given by

$$T = \begin{bmatrix} I_8 & 0 & 0 \\ 0 & \Gamma & 0 \\ 0 & 0 & \Gamma^{-1} \\ 0 & 0 & 0 \end{bmatrix}$$

where Γ is the three by three direction cosine matrix relating the angular orientation of the body-mounted coordinate system with respect to the navigation coordinate frame.

$$\Gamma = \begin{bmatrix} C_{x'x} & C_{x'y} & C_{x'z} \\ C_{y'x} & C_{y'y} & C_{y'z} \\ C_{z'x} & C_{z'y} & C_{z'z} \end{bmatrix}$$

The initial estimation error covariance matrix to be used in the filters is given, in the x'y'z' coordinate system, by

$$P(0|0) = TP^*(0|0)T'.$$

The variances and inverse time constants used to characterize the random processes in the system model are as follows:

$$\begin{aligned} \text{accelerometer biases: } \sigma^2 &= 400 \text{ sec}^2 \\ \beta &= 1/15 \text{ hr}^{-1} \end{aligned}$$

gyro biases: $\sigma^2 = 0.09 \left(\frac{\text{deg}}{\text{hr}} \right)^2$

$$\beta = 1/15 \text{ hr}^{-1}$$

initialized at $\sigma = 0.03 \text{ deg/hr}$

vertical velocity error: $\sigma^2 = 0.25 \left(\frac{\text{ft}}{\text{sec}} \right)^2$

$$\beta = 0.18 \text{ sec}^{-1}$$

altitude error: initialized at $\sigma = 200 \text{ ft}$

doppler count bias: $\sigma^2 = 10^6$

$$\beta = 10^{-7} \text{ sec}^{-1}$$

white noise constants: $u_1^2 = u_2^2 = u_3^2 = 1 \text{ min}^2/\text{hr}$

$$u_5^2 = u_7^2 = 1 \text{ knot}^2/\text{hr}$$

The measurement noise covariance matrix $V(k)$ was set to the constant value of 1600.

To compare the performance of the different filters, true error curves have been plotted with the estimates. The true error curves were obtained by tracking the vehicle from ground stations. Due to the proprietary nature of this data, the plots have been normalized.

E. Design Model Based on a Simplified Direction Cosine Matrix

In order for the A matrix in the system model to be evaluated at each stage, the direction cosine matrix relating the body-mounted coordinate system must be updated at each stage. Under the assumption of level flight, the computation of this matrix becomes considerably simplified. We discuss here the results of using a design system model based on this assumption.

Define the roll angle α as the rotation of the body-mounted system about the x axis, the pitch angle β as the rotation of the body-mounted system about the -y axis, and the yaw angle γ as the rotation of the body-mounted system about the -z axis. The direction cosine matrix Γ^{-1} relating the body-mounted system to the navigational system becomes (see, e.g., Pitman [23])

$$\Gamma^{-1} = \begin{bmatrix} c\alpha c\gamma & c\alpha s\gamma - s\alpha s\beta c\gamma & -s\alpha s\gamma - c\alpha s\beta c\gamma \\ -c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -s\alpha c\gamma \\ s\beta & s\alpha c\beta & c\alpha s\beta s\gamma \end{bmatrix}$$

where c and s indicate the cosine and sine of the indicated angle, respectively.

If level flight is assumed, the only angular displacement which occurs between the body-mounted system and the navigational system is in yaw. The resulting simplified direction cosine matrix is

$$\Gamma_d^{-1} = \begin{bmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using Γ_d^{-1} in the evaluation of the A matrix results in a design model which is used in the delayed-state filter as in chapter 3, section A. We call this model design model 1.

Figure 6 shows the normalized estimates of the latitude error of the optimal filter and the suboptimal filter resulting from this design model as a function of the flight time in Greenwich mean time. For comparison, the normalized true latitude error curve is shown also. Similarly, the longitude error estimates are shown in Fig. 7.

Upon comparison of Fig. 6 and 7, we see little change in the filter's performance is introduced by the assumption of level flight. This result was found to be true for the estimates of the other state variables also.

The design model resulting from this simplification is by itself not much simpler than the original model. However, this simplification would probably be made in conjunction with other modifications in the original model such as those to be discussed in the next section. To assess the effect of each change in the model on the filter performance, the changes must be made one at a time in order to determine which one is at fault if poor performance results.

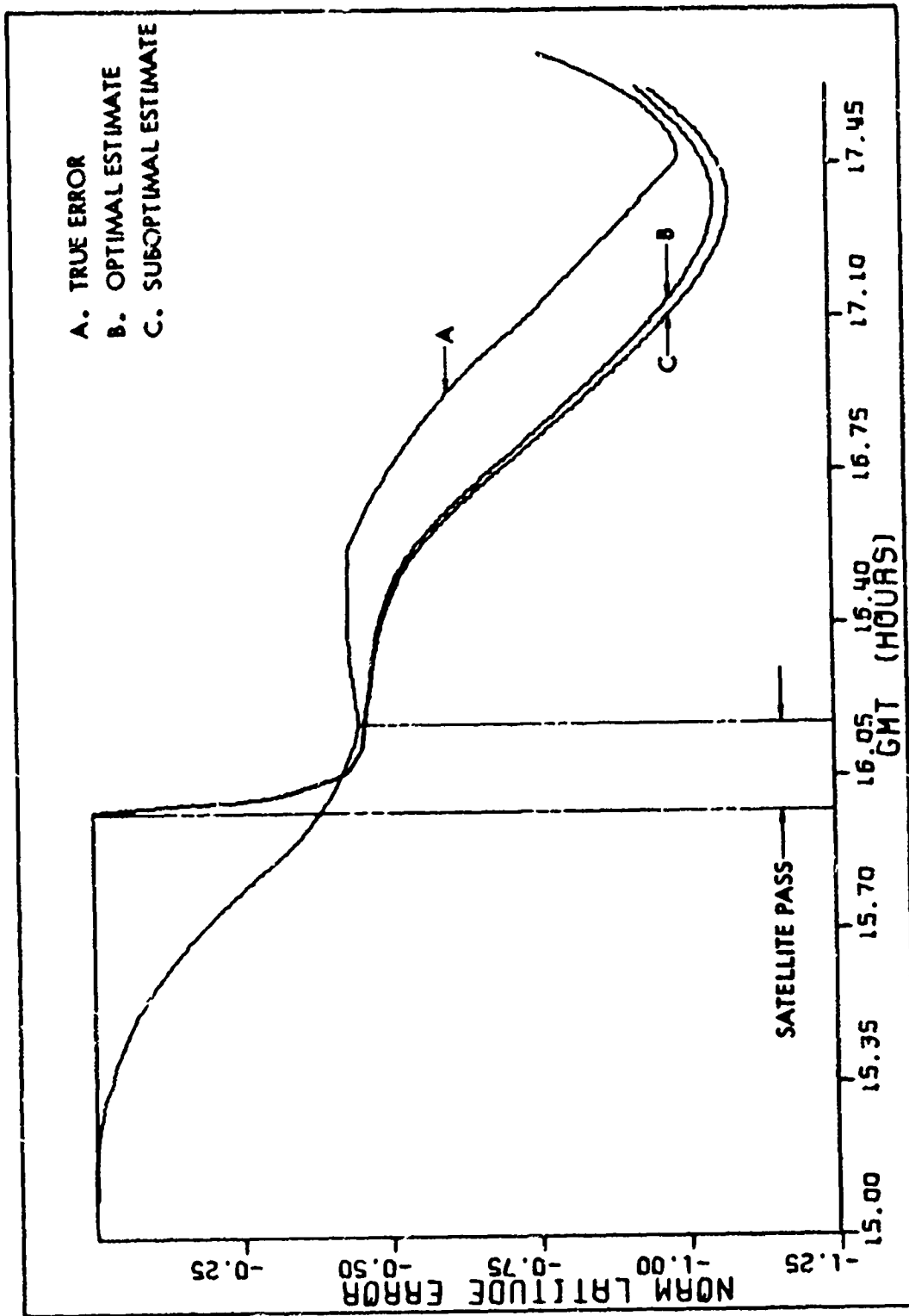


Fig. 6. Design model 1, normalized latitude error

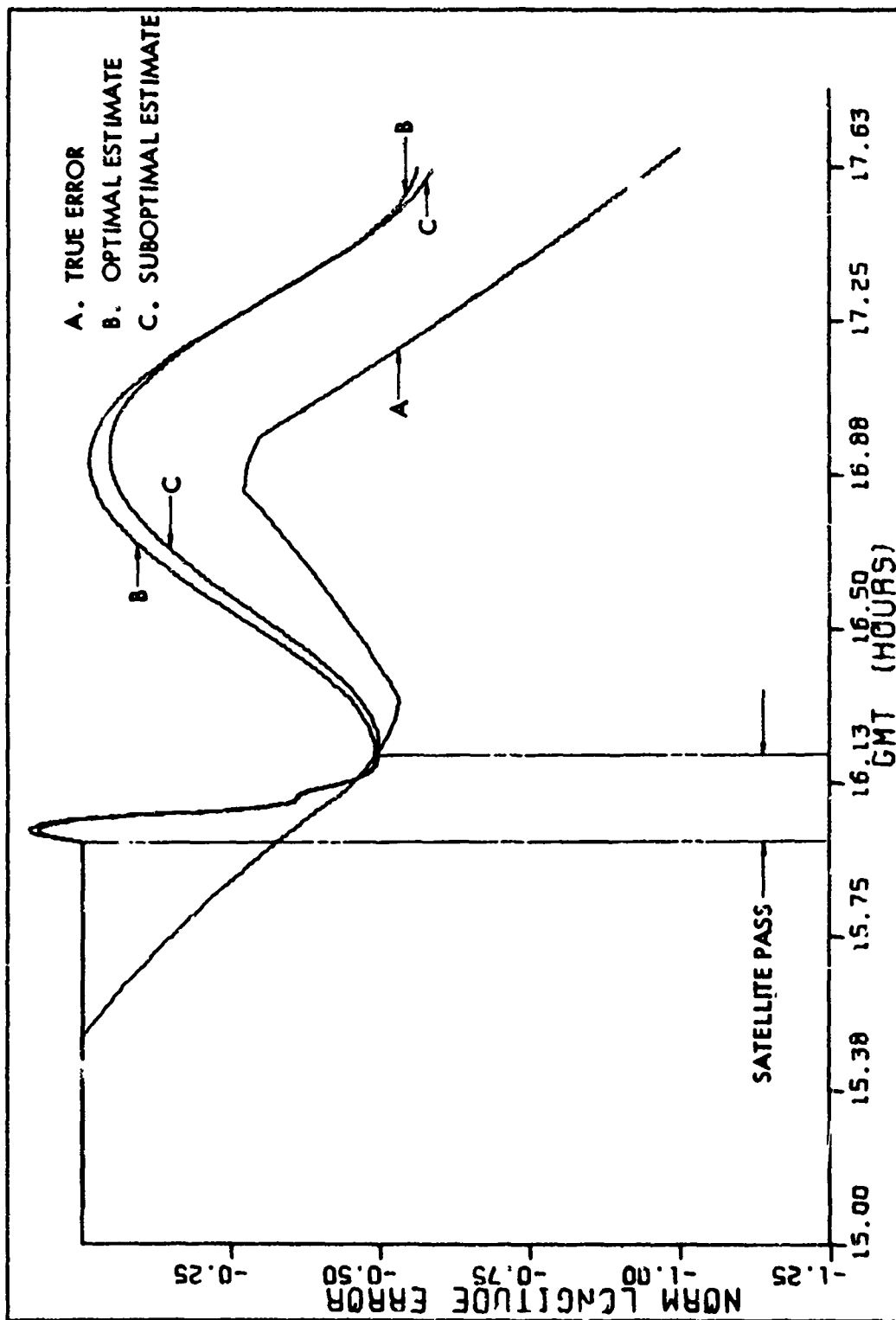


Fig. 7. Design model 1, normalized longitude error

F. Design Model Based on Simplified Dynamics in the Level Channels

We present here the results of using a design model based upon simplification of the dynamical equations for the error in the two level channels.

The basic equation used to derive the two level channel equations for the original model is equation (5.2).

$$\ddot{\delta R} + 2\omega \times \dot{\delta R} + \dot{\omega} \times \delta R + \omega \times (\omega \times \delta R) = \delta a - \psi \times a - \omega_o^2 R_{\tan}.$$

A number of simplifying assumptions on this equation seem reasonable due to the nature of the vehicle and typical flights.

If $\dot{\omega}$ is set equal to zero ($\Delta\omega \ll \omega_o$ in a Δt interval), the centripetal acceleration is neglected, and R is considered to remain constant ($\Delta R \ll R$ in a Δt interval), then equation (5.2) becomes

$$\ddot{\delta R} + 2\omega \times \dot{\delta R} = \delta a - \psi \times a - \omega_o^2 R_{\tan}$$

and the two level channel equations become

$$\dot{x}_5 = -\omega_o x_1 - \omega_o x_4 + 2\omega_z x_7 - \omega_o [C_{yx} x_{12} + C_{yy} x_{13} + C_{yz} x_{14}] - 2\omega_z x_{15}$$

$$\dot{x}_7 = -\omega_o x_2 - 2\omega_z x_5 - \omega_o x_6 + \omega_o [C_{xx} x_{12} + C_{xy} x_{13} + C_{xz} x_{14}] - 2\omega_y x_{15}.$$

By replacing the elements of the fifth and seventh rows of the A matrix of the original model with the coefficients of the above equations we obtain the design model A matrix. We call the resulting model design model 2.

The results of using design model 2 in the filter are shown in Fig. 8 and Fig. 9. There is little significant difference in the estimates of both the latitude and longitude error. In fact, the suboptimal estimates appear to be a little better than the optimal ones. At first thought, this would be cause for concern, until it is noted that these results are only one sample out of a possibly large ensemble of samples. The only conclusion to be drawn is that the above assumptions do not significantly alter the performance of the filter, at least for flight trajectories similar to the one simulated.

The results of incorporating the simplifications of design model 1 and design model 2 in one model are shown in Figs. 10 and 11. The effect was nearly the same as using only one or the other of the simplifications; in fact, the performance was slightly better in the longitude channel.

With efficient programming, a significant reduction can be made in the computational effort required to compute the transition matrix without significant degradation in the filter's performance by incorporating the simplifications of design models 1 and 2. This is important because the transition matrix must be computed at the end of each 20-second interval throughout the duration of a flight.

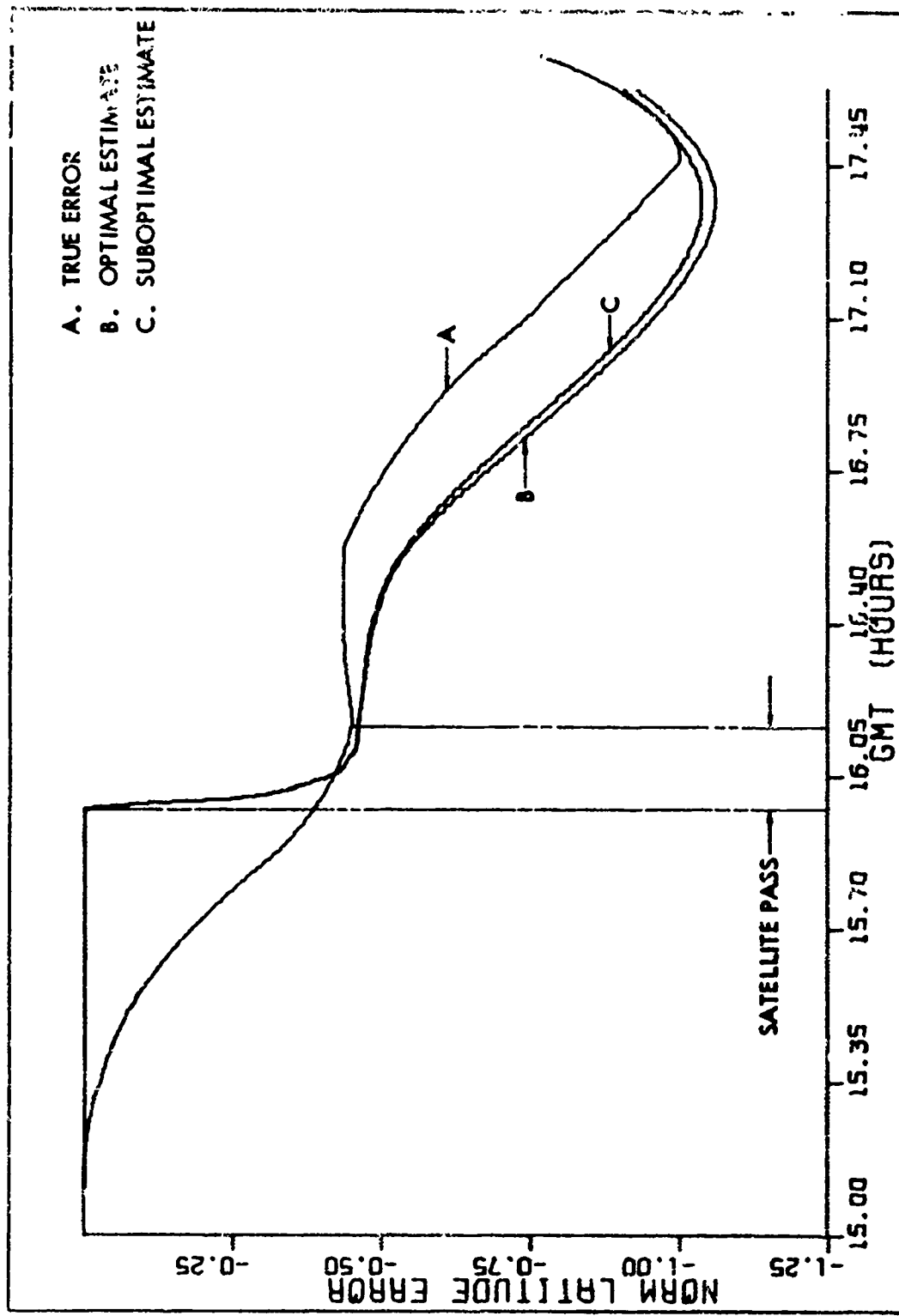


Fig. 8. Design model 2, normalized latitude error

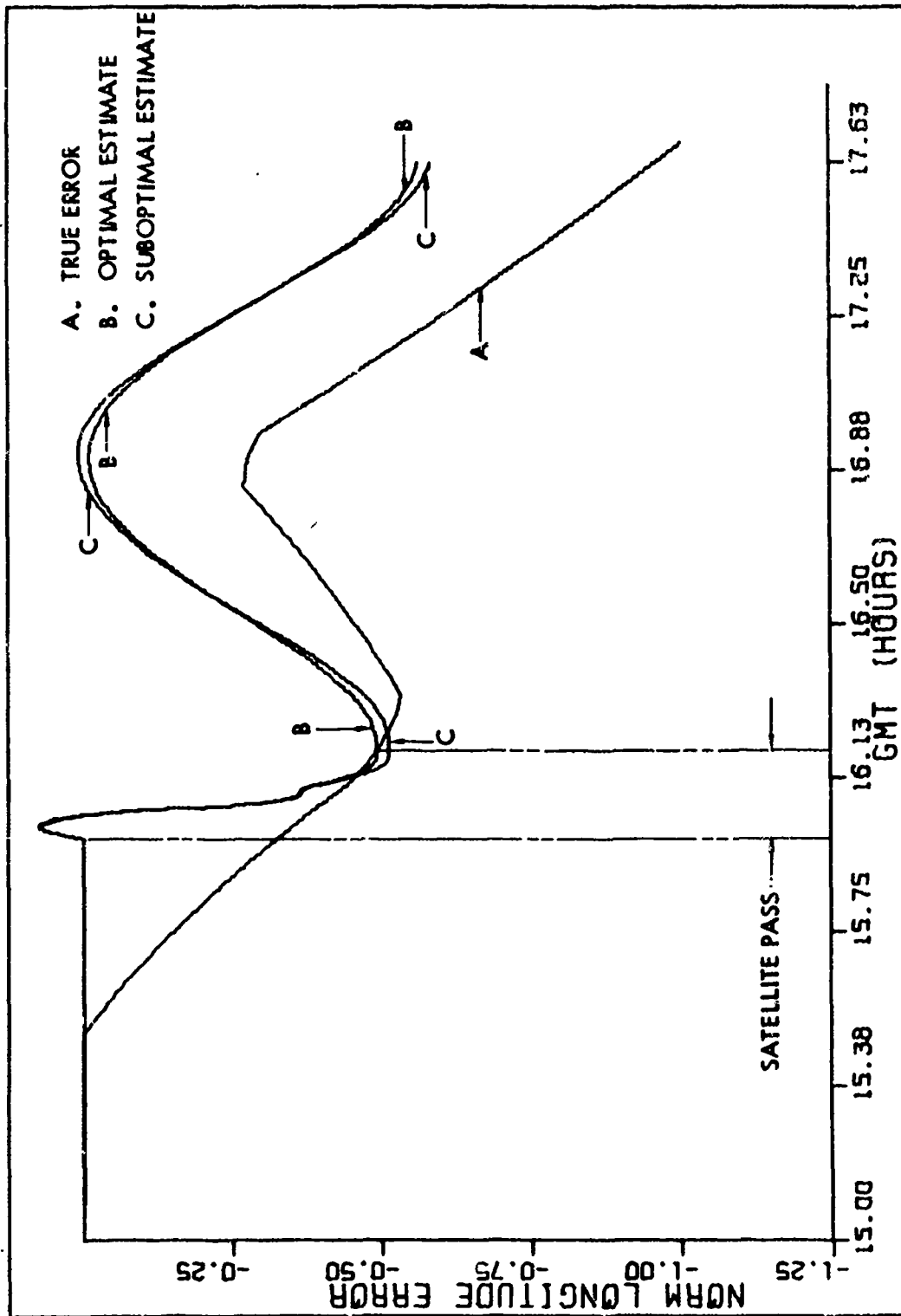


Fig. 9. Design model 2, normalized longitude error

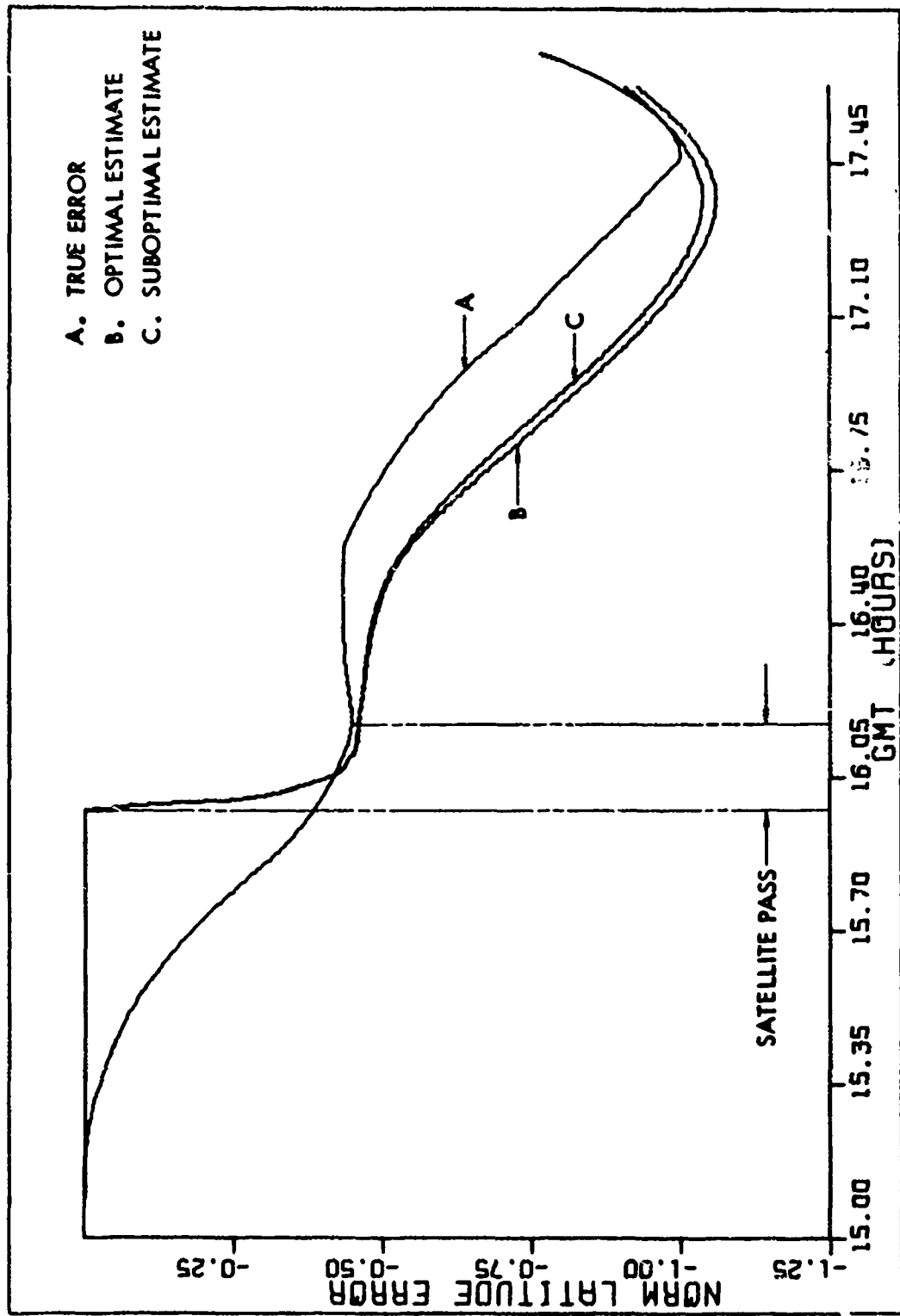


Fig. 10. Design models 1 and 2 combined, normalized latitude error

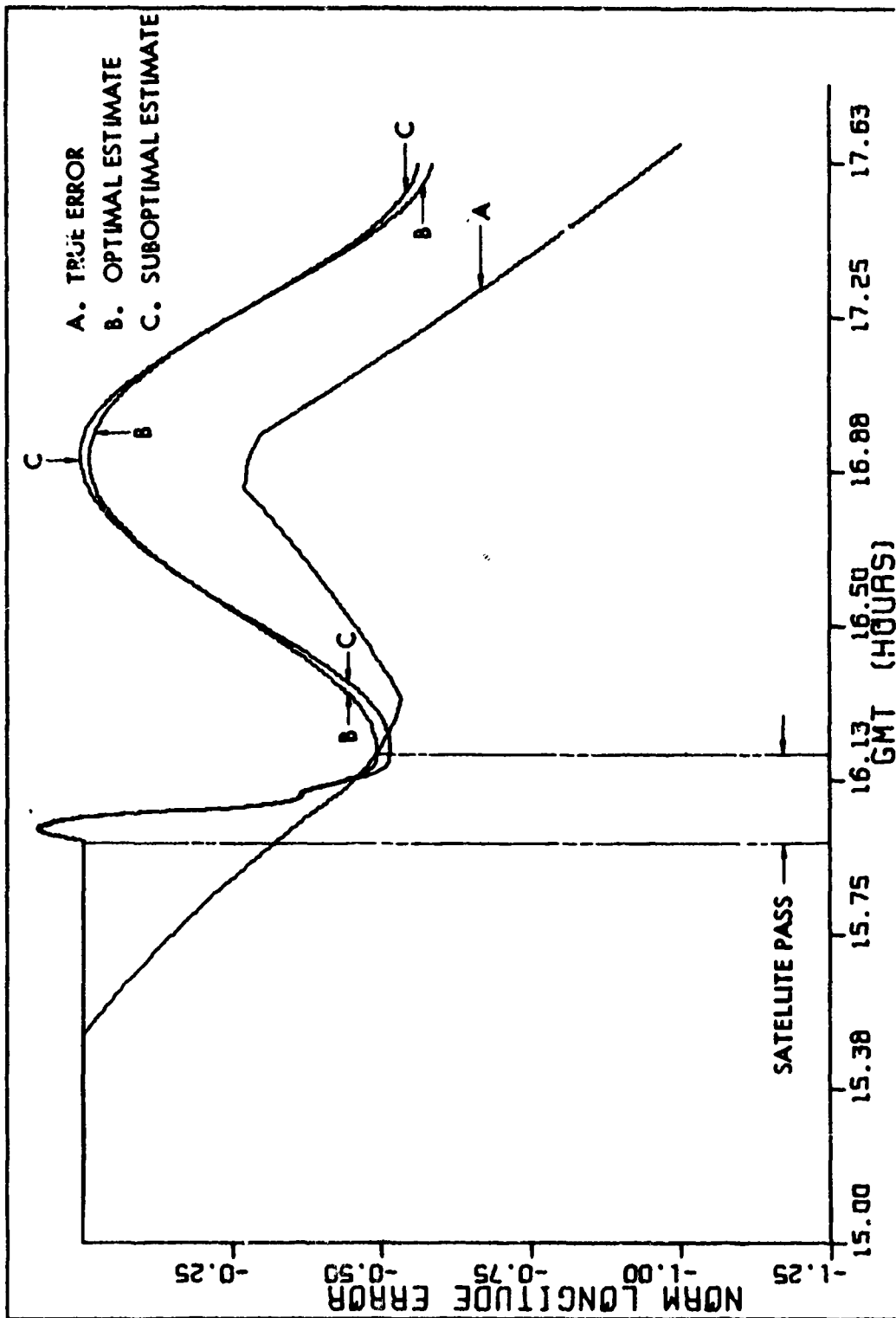


Fig. 11. Design models 1 and 2 combined, normalized longitude error

G. Suboptimization Using the Partitioning Method

Attempts were made to apply the partitioning method of chapter IV to the above system for two choices of partitions. Although achieving a considerable reduction in the computational requirements, both choices resulted in unsatisfactory performance of the filter. Both the estimator and the estimation error covariance matrix equations diverged rapidly from the optimal filter results by several orders of magnitude after the satellite pass period. During the pass period when measurement data is available, transitory oscillations were observed.

In the first partitioning that was tried, the system was partitioned into three subsystems. Subsystem one contained x_1 , x_4 , and x_5 , i.e., level channel one and the coupled psi variable. Subsystem two contained x_2 , x_6 , and x_7 , i.e., level channel two and the coupled psi variable. The third subsystem contained x_1 , x_2 , x_3 , x_8 , . . . , x_{16} , i.e., the twenty-four hour dynamics, the altitude channel, and the first order Markov instrument noise processes. The measurement y was used in all three subsystems.

The second partitioning tried consisted of two subsystems. The first subsystem contained the first eight state variables and x_{15} and x_{16} . The second subsystem contained x_9 , . . . , x_{14} , the instrument noise processes. The measurement y was used in both subsystems. This choice is equivalent to modeling the gyro biases and accelerometer biases as white noise. Due to the form of the measurement matrices M and N , the second subsystem

state variables are estimated as the initial estimate projected through the transition matrix.

The divergence problem may be due to one or all of several factors. First, the nature of the form of measurement for this system may not be well suited to suboptimization by this method. Information about sixteen state variables must be derived from a scalar measurement. This raises the question of whether or not the observability properties of the system place a constraint on the choice of partitioning or, more fundamentally, whether or not the partitioning method is applicable to the system at all. However, both the observability of the above system and the relationship of observability with the partitioning method remain unexplored problems.

A second important consideration is the fact that only two partitionings were tried. It is believed that the successful application of the method to this system is highly dependent on the proper choice of partitioning.

A third factor, although not a dominant one, to consider is the error due to the digital computations. In any study of this sort, truncation error and other numerical errors will occur to some degree. The simulation of this system is particularly vulnerable since many of the matrices involved are ill-conditioned and a great number of matrix operations are required.

As a final consideration, in the derivation of the partitioning method, the pseudo-inverse of the C_j 's was used to approximate the total system state in order to de-couple the subsystem model from the total

system model. For many cases this may be a very poor approximation. For example, the first two subsystems of the first partitioning are only three-dimensional and the total system is sixteen-dimensional. Three of the total system state variables are approximated with the subsystem state variables and the remaining are approximated as zero for all time. As indicated above, it may be possible to improve this approximation by a different choice of partitioning.

VI. CONCLUSIONS

Many attempts to solve the excessive computational problem of the Kalman filter may be categorized into two classes of suboptimizations. The first class may be characterized by a simplification in the system model that is used in the filter equations. The second class takes advantage of a suboptimal gain computation with the other filter equations remaining the same as for the optimal filter. In chapter III, we presented a performance analysis for both of the above classes of suboptimization for the delayed-state Kalman filter. The derivation of a suboptimization of the delayed-state filter, which belongs to the second class, was presented in chapter IV.

In chapter V, the results of a simulation study of the performance of an integrated inertial/doppler-satellite navigation system were presented. It was found that for two types of model simplifications the suboptimal filter performed very well in comparison to the optimal filter. On the other hand, attempts to apply the results of chapter IV to that system were not very successful.

Several problems remain unsolved. Further experimentation might result in a partitioning which results in a satisfactory performance by the partitioned suboptimal filter for the above example.

A simulation of the performance analysis equations and use of the performance index was not attempted. However, the performance index was computed for the partitioned filter and confirmed the other indications that the filter was not performing satisfactorily. Concerted

effort may result in the reduction of the performance analysis equations for modeling changes to a simpler and more practical computational form.

Optimization of the partitioned filter by the choice of the F_j matrices with respect to the mean-squared estimation error appears possible. However, this would result in r additional recursive equations. Their dimensionality would be less than that of the total system state and the amount of total computation required should still be less than that required by the optimal filter. The partitioned filter would still be suboptimal with respect to the optimal one, but should have improved performance over the partitioned filter presented in chapter IV.

VII. ACKNOWLEDGMENTS

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VIII. APPENDIX:

DERIVATION OF A KALMAN FILTER WITH DELAYED-STATES AS OBSERVABLES

We present here a derivation of the recursive equations for the optimal delayed-state Kalman filter. The results of which were first obtained through the work of Brown and Hartman [7]. Our method is not the only approach that could be used. The derivation is presented merely to make more rigorous and complete the results in the preceding chapters.

The optimality of the filter is a consequence of a fundamental theorem in estimation theory due to Sherman [26] which we will state without proof (see Deutsch [8]). We will then state a more specialized result which will be used in our derivation.

Let $x(k)$ be an n -dimensional random variable with m -dimensional measurements $y(1), \dots, y(j)$. The conditional probability distribution function of $x(k)$ given $y(1), \dots, y(k)$ is

$$F[\xi | y(1), \dots, y(j)] = P[x(k) \leq \xi | z(j)]$$

where P is a suitable probability measure and $z(j)$ is an mj -dimensional vector whose components are $y_1(1), \dots, y_m(1), \dots, y_1(j), \dots, y_m(j)$.

We wish to estimate $x(k)$ based only on the measurements $y(1), \dots, y(j)$. We denote this estimate by $\hat{x}(k|j)$. We are concerned here only with

the filter problem when $j = k$. We have the smoothing problem when $j > k$ and the prediction problem when $j < k$.

We denote the estimation error by

$$\tilde{x}(k|j) = x(k) - \hat{x}(k|j).$$

We would like $\tilde{x}(k|j)$ to be zero. When it is not, we assign a penalty or loss for an incorrect estimate. We do this by specifying an admissible loss function $L[\tilde{x}(k|j)]$ which has the following properties:

1. L is a non-negative function from R^n to R .
2. $L(0) = 0$, where the first zero denotes the zero n -vector.
3. $L[\tilde{x}_b(k|j)] \geq L[\tilde{x}_a(k|j)]$, whenever

$$g[\tilde{x}_b(k|j)] \geq g[\tilde{x}_a(k|j)], \text{ where } g \text{ is a non-negative convex}$$

function from R^n to R .

4. $L[\tilde{x}(k|j)] = L[-\tilde{x}(k|j)]$.

An estimate is optimal if it minimizes L .

We now state the fundamental theorem.

THEOREM 1.

If L is an admissible loss function and $F[\xi|z(j)]$ is:

- a. Symmetric about its mean
- b. Convex for all $\xi \in E\{\xi\}$

then the optimal estimate is

$$\hat{x}(k|j) = E\{x(k) | z(j)\}.$$

The following theorem, which we also state without proof (see, e.g., Meditch [18] or Kalman [13]), is a direct consequence of Theorem 1. It is this result upon which the Kalman filter equations are based.

THEOREM 2.

If only the first and second moments of the stochastic processes $\{x(k), k = 0, 1, \dots\}$ and $\{y(i), i = 1, 2, \dots, j\}$ are known, then the optimal estimate for all admissible loss functions is the linear estimate

$$\hat{x}(k|j) = E\{x(k)\} + P_{xz} P_{zz}^{-1} \{z(j) - E[z(j)]\}$$

where P_{xz} is the $n \times m_j$ cross-covariance matrix of $x(k)$ and $z(j)$ and P_{zz} is the covariance matrix of $z(j)$.

The model of our dynamical system was given in chapter II. We repeat it here for convenience.

$$x(k+1) = \Phi(k+1, k)x(k) + u(k) \quad (\text{A.1})$$

$$y(k+1) = M(k+1)x(k+1) + N(k+1)x(k) + v(k+1) \quad (\text{A.2})$$

The assumptions placed upon this model in chapter II are required here also.

From Theorem 1, the optimal estimate at $k+1$ given measurements $y(1), \dots, y(k+1)$ is

$$\hat{x}(k+1|k+1) = E\{x(k+1) | z(k+1)\} \quad (\text{A.3})$$

For any gaussian zero-mean random variables x, y, z (Papoulis [21] or Meditch [18])

$$E\{x|y, z\} = E\{x|y, \tilde{z}\},$$

where $\tilde{z} = z - E\{z|y\}$, and

$$E\{x|y, \tilde{z}\} = E\{x|y\} + E\{x|\tilde{z}\}.$$

Applying these formulas to (A.3), we find that

$$\begin{aligned}\hat{x}(k+1|k+1) &= E\{x(k+1)|z(k)\} + E\{x(k+1)|\tilde{y}(k+1|k)\} \\ &= \hat{x}(k+1|k) + E\{x(k+1)|\tilde{y}(k+1|k)\}\end{aligned}\quad (A.4)$$

where

$$\tilde{y}(k+1|k) = y(k+1) - E\{y(k+1)|z(k)\}.$$

Letting $\hat{y}(k+1|k)$ be the optimal estimate of the measurement $y(k+1)$ given measurements through time k , we have $\tilde{y}(k+1|k) = y(k+1) - \hat{y}(k+1|k)$. Expanding $\hat{y}(k+1|k)$, noting that $v(k+1)$ is independent of $\{y(1), \dots, y(k)\}$.

$$\begin{aligned}\hat{y}(k+1|k) &= E\{M(k+1)x(k+1) + N(k+1)x(k) + v(k+1)|z(k)\} \\ &= M(k+1)\hat{x}(k+1|k) + N(k+1)\hat{x}(k|k)\end{aligned}\quad (A.5)$$

Noting that $\hat{x}(k+1|k) = \Phi(k+1, k)\hat{x}(k|k)$, equation (A.4) becomes

$$\hat{x}(k+1|k+1) = \Phi(k+1, k)\hat{x}(k|k) + E\{x(k+1)|\tilde{y}(k+1|k)\}. \quad (A.6)$$

Since $x(k+1)$ and $\tilde{y}(k+1|k)$ are zero mean and gaussian, we have as a consequence of Theorem 2

$$E\{x(k+1)|\tilde{y}(k+1|k)\} = P_{xy}P_{yy}^{-1}\tilde{y}(k+1|k) \quad (A.7)$$

where P_{xy} is the cross-covariance matrix of $x(k+1)$ and $\tilde{y}(k+1|k)$ and

$P_{\tilde{y}\tilde{y}}$ is the covariance matrix of the measurement estimation error $\tilde{y}(k+1|k)$; i.e.,

$$P_{x\tilde{y}} = E\left\{x(k+1)\tilde{y}'(k+1|k)\right\}$$

$$P_{\tilde{y}\tilde{y}} = E\left\{y(k+1|k)\tilde{y}'(k+1|k)\right\}$$

Defining $K(k+1) = P_{x\tilde{y}}P_{\tilde{y}\tilde{y}}^{-1}$, we have

$$\begin{aligned} E\left\{x(k+1)|y(k+1|k)\right\} &= K(k+1) [y(k+1) - M(k+1)\Phi(k+1,k)\hat{x}(k|k) \\ &\quad - N(k+1)\hat{x}(k|k)] \\ &= K(k+1)[M(k+1)\Phi(k+1,k)x(k) + N(k+1)x(k) \\ &\quad + v(k+1) - M(k+1)\Phi(k+1,k)\hat{x}(k|k) \\ &\quad - N(k+1)\hat{x}(k|k)] \\ &= K(k+1)[M(k+1)\Phi(k+1,k) + N(k+1)]\tilde{x}(k|k) \\ &\quad + K(k+1)v(k+1). \end{aligned} \quad (A.8)$$

Thus, the form of equation (A.6) which corresponds to equation (2.7) in chapter II is

$$\begin{aligned} \hat{x}(k+1|k+1) &= \hat{x}(k+1) + K(k+1)[y(k+1) - M(k+1)\hat{x}(k+1|k) \\ &\quad - N(k+1)\hat{x}(k|k)] . \end{aligned} \quad (A.9)$$

To evaluate $K(k+1)$,

$$\begin{aligned} P_{x\tilde{y}} &= E\left\{x(k+1)\tilde{y}'(k+1|k)\right\} \\ &= E\left\{[\tilde{x}(k+1|k) - \hat{x}(k+1|k)][M(k+1)\tilde{x}(k+1|k) \right. \\ &\quad \left. + N(k+1)\tilde{x}(k|k) + v(k+1)]'\right\} \\ &= P(k+1|k)M'(k+1) + \Phi(k+1,k)P(k|k)N'(k+1) \end{aligned} \quad (A.10)$$

where we have used the fact that $v(k+1)$ is independent of $x(k|k)$ and $P(k+1|k)$ by definition is $E\left\{x(k+1|k)x(k+1|k)'\right\}$.

Evaluating \tilde{P}_{yy} ,

$$\begin{aligned}
 \tilde{P}_{yy} &= E\left\{\tilde{y}(k+1|k)y'(k+1|k)\right\} \\
 &= E\left\{[M(k+1)\tilde{x}(k+1|k) + N(k+1)\tilde{x}(k|k) + v(k+1)][M(k+1)\tilde{x}(k+1|k) \right. \\
 &\quad \left. + N(k+1)\tilde{x}(k|k) + v(k+1)]'\right\} \\
 &= M(k+1)P(k+1|k)M'(k+1) + V(k+1) + N(k+1)P(k|k)N'(k+1) \\
 &\quad + M(k+1)\Phi(k+1,k)P(k|k)N'(k+1) \\
 &\quad + [M(k+1)\Phi(k+1,k)P(k|k)N'(k+1)]' \quad (A.11)
 \end{aligned}$$

Upon comparing (A.11) with equation (2.4) we see that \tilde{P}_{yy} is $Q(k+1)$.

Thus, the optimal gain matrix $K(k+1)$ is given by

$$K(k+1) = [P(k+1|k)M'(k+1) + \Phi(k+1,k)P(k|k)N'(k+1)]Q^{-1}(k+1) \quad (A.12)$$

where $Q(k+1)$ is given by (A.11). The inverse will always exist since $V(k+1)$ is assumed positive definite.

$P(k+1|k)$ is given by

$$\begin{aligned}
 P(k+1|k) &= E\left\{\tilde{x}(k+1|k)\tilde{x}'(k+1|k)\right\} \\
 &= \Phi(k+1,k)P(k|k)\Phi'(k+1,k) + H(k) \quad (A.13)
 \end{aligned}$$

Expanding $\tilde{x}(k+1|k+1)$, we obtain

$$\begin{aligned}
 \tilde{x}(k+1|k+1) &= [I - K(k+1)M(k+1)]\tilde{x}(k+1|k) - K(k+1)N(k+1)\tilde{x}(k|k) \\
 &\quad - K(k+1)v(k+1).
 \end{aligned}$$

Noting that $E\left\{\tilde{x}(k+1|k)\tilde{x}'(k|k)\right\} = \Phi(k+1,k)P(k|k)$,

$$\begin{aligned}
P(k+1|k+1) &= E\left\{\tilde{x}(k+1|k+1)\tilde{x}'(k+1|k+1)\right\} \\
&= P(k+1|k) - K(k+1)M(k+1)P(k+1|k) \\
&\quad - [K(k+1)M(k+1)P(k+1|k)]' \\
&\quad - K(k+1)N(k+1)P(k|k)\Phi'(k+1,k) \\
&\quad - [K(k+1)N(k+1)P(k|k)\Phi'(k+1,k)]' \\
&\quad + K(k+1)[M(k+1)P(k+1|k)M'(k+1) \\
&\quad + M(k+1)\Phi(k+1,k)P(k|k)N'(k+1) \\
&\quad + \{M(k+1)\Phi(k+1,k)P(k|k)N'(k+1)\}]' \\
&\quad + N(k+1)P(k|k)N'(k+1) + V(k+1)]K'(k+1) \quad (A.14)
\end{aligned}$$

After some manipulation of (A.14) and the use of equation (A.13),

$$P(k+1|k+1) = P(k+1|k) - K(k+1)Q(k+1)K'(k+1). \quad (A.15)$$

Note that $\tilde{x}(k|k)$ is a gaussian zero mean process and, hence,

$$P(0|0) = E\{x(0)x'(0)\} \text{ since } \hat{x}(0|0) = E\{x(0)\} = 0.$$

This completes the derivation of the optimal delayed-state filter.

The data, beginning at time $k = 1$, are processed by cycling through the recursive equations (A.13), (A.11), (A.12), (A.15), and (A.9) with initial conditions $P(0|0) = E\{x(0)x'(0)\}$ and $\hat{x}(0|0) = 0$.

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<p>The computational requirements of the Kalman filter may become excessive when the measurement model includes a connection with both the present state and a previous state. Three aspects of this problem are studied. A performance analysis is presented and a performance index is defined as an aid in evaluating the performance of two classes of suboptimal filters which may be used to solve this problem. The two classes are suboptimality due to modeling variations and due to alternate gain algorithms. A suboptimal filter is derived which belongs to the second class. The simulation of a proposed integrated inertial/doppler-satellite navigation system is performed to study the performance of filters belonging to both of the above classes.</p>		

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